The isomorphism problem of Hausdorff measures and Hölder restrictions of functions

Theses of the doctoral thesis

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1 Introduction

Geometric measure theory is concerned with investigating subsets of Euclidean spaces from measure theoretical point of view. Its basic tool is the Hausdorff measure. As it is well-known, for every $0 \le s \le n$ there exists a measure $\mathcal{H}^s$, the so-called s-dimensional Hausdorff measure on $\mathbb{R}^n$. Therefore, it is a fundamental question whether Hausdorff measures of different dimensions can be Borel isomorphic. This was an unsolved problem for several years, attributed to B. Weiss and popularized by D. Preiss. Let $\mathcal{B}$ denote the σ-algebra of the Borel subsets of $\mathbb{R}$. Then the exact question reads as follows.

Question 1. Let $0 \le s, t \le 1$ and $s \ne t$.

(i) Can the measure spaces $(\mathbb{R}, \mathcal{B}, \mathcal{H}^s)$ and $(\mathbb{R}, \mathcal{B}, \mathcal{H}^t)$ be isomorphic?

(ii) Does there exist a Borel bijection $f : \mathbb{R} \to \mathbb{R}$ such that for every Borel set B,

$$0 < \mathcal{H}^s(B) < \infty \Longleftrightarrow 0 < \mathcal{H}^t(f(B)) < \infty?$$

The two parts are not equivalent but it is easy to see that a negative answer to (ii) implies a negative answer to (i).

It is important to make the distinction that we are looking for *Borel* isomorphisms only. M. Elekes [4] proved that if we assume the continuum hypothesis then the measure spaces $(\mathbb{R}, \mathcal{M}_{\mathcal{H}^s}, \mathcal{H}^s)$ and $(\mathbb{R}, \mathcal{M}_{\mathcal{H}^t}, \mathcal{H}^t)$ are isomorphic whenever $s, t \in (0, 1)$, where $\mathcal{M}_{\mathcal{H}^d}$ denotes the σ-algebra of the sets which are measurable with respect to $\mathcal{H}^d$. However, such an isomorphism need not be Borel.

M. Elekes, aiming to solve the original problem (Question 1), raised the following question [4].

Question 2. Let $0 < \alpha < 1$. Can we find for every Borel (or continuous, or typical continuous) function $f : [0, 1] \to \mathbb{R}$ a Borel set $B \subset [0, 1]$ of positive Hausdorff dimension such that f restricted to B is Hölder continuous of exponent α?

How is this question related to the previous one? Suppose that we have an answer to Question 2 saying that (for some fixed α) for every Borel function f there exists a Borel set B of dimension β such that f is Hölder-α on B. As it is well-known, this implies that $f(B)$ has dimension at most β/α. It is easy to see that this would answer (both parts of) Question 1 in the negative for those s and t for which $0 \le s < \beta < \beta/\alpha < t \le 1$ holds.

According to a theorem of P. Humke and M. Laczkovich [5], a typical continuous function $f : [0, 1] \to \mathbb{R}$ is not monotonic on any set of positive Hausdorff dimension.

Since every function of bounded variation is the sum of two monotonic functions, this theorem motivated M. Elekes to raise an analogue of Question 2.

Question 3. Can we find for every Borel (or continuous, or typical continuous) function $f : [0,1] \to \mathbb{R}$ a Borel set $B \subset [0,1]$ of positive Hausdorff dimension such that f restricted to B is of bounded variation? Can we even find such a set of dimension $1/2$?

This problem has also been circulated by D. Preiss, and a similar question was already asked by P. Humke and M. Laczkovich, see also Z. Buczolich [2, 3].

Questions 2 and 3 belong to the topic of *restriction theorems*. Usually the setting of a restriction theorem is the following. Given some function f from some class X, one tries to find a large set A such that $f|_A$ belongs to some other (nice) class Y. Here largeness usually means that A is infinite, uncountable, perfect, not porous, or A is of positive measure or of second category. We refer to the survey article of J. B. Brown [1] on restriction theorems and to the references therein.

M. Elekes gave partial answers to Question 3 and Question 2 in [4]. He proved that a typical continuous function $f : [0,1] \to \mathbb{R}$ is not of bounded variation on any set of Hausdorff dimension larger than $1/2$. Regarding Question 2, he also gave an upper bound for the possible dimension by showing that for every $0 < \alpha \leq 1$, a typical continuous function is not Hölder-α on any set of dimension larger than $1 - \alpha$.

2 Results

2.1 Restrictions of functions

We answer Question 2 and Question 3 by proving the following theorems.

Theorem 2.1.1. *Let $f : [0,1] \to \mathbb{R}$ be Lebesgue measurable. Then there exists a compact set $C \subset [0,1]$ of Hausdorff dimension $1/2$ such that $f|_C$ is of bounded variation.*

Theorem 2.1.2. *Let $f : [0,1] \to \mathbb{R}$ be Lebesgue measurable and let $0 < \alpha < 1$. Then there exists a compact set $C \subset [0,1]$ of Hausdorff dimension $1 - \alpha$ such that $f|_C$ is a Hölder-α function.*

That is, the dimension bounds found by M. Elekes are sharp.

To prove Theorem 2.1.1, first we define discrete Hausdorff pre-measures on $\mathbb{Z}$ (the integers). Using this notion we are able to formalize and solve a discrete (quantitative) version of the problem. Then suitable limit theorems yield Theorem 2.1.1. It is possible to prove Theorem 2.1.2 in exactly the same way. However, we present a simple proof instead, using a theorem of P. Mattila about Hausdorff dimensions of plane sections.

We also discuss a variant of Theorem 2.1.1 regarding generalized variations, and we extend our theorems to Euclidean spaces.

These results can essentially be found in [7].

2.2 Borel maps and Hausdorff dimension

As explained in the Introduction, Theorem 2.1.2 gives some partial results on the isomorphism problem of Hausdorff measures (Question 1). For example, applying the theorem for $\alpha = 2/3$ implies that the Hausdorff measures of dimension $1/3 - \varepsilon$ and $1/2 + \varepsilon$ cannot be Borel isomorphic. However, this approach does not seem to answer Question 1 in its full generality. We prove the following theorem.

Theorem 3.1.2. *Let $0 \leq d \leq 1$ and let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable. Then there exists a compact set C of Hausdorff dimension d such that $f(C)$ is of Hausdorff dimension at most d.*

Clearly, this theorem immediately implies that the Hausdorff measures of different dimensions cannot be Borel isomorphic; that is, we answer (both parts of) Question 1 in the negative.

Theorem 3.1.1. *For every $0 \leq s < t \leq 1$ the measure spaces $(\mathbb{R}, \mathcal{B}, \mathcal{H}^s)$ and $(\mathbb{R}, \mathcal{B}, \mathcal{H}^t)$ are not isomorphic. Moreover, there does not exist a Borel bijection $f : \mathbb{R} \to \mathbb{R}$ such that for every Borel set $B \subset \mathbb{R}$,*

$$0 < \mathcal{H}^s(B) < \infty \Longleftrightarrow 0 < \mathcal{H}^t(f(B)) < \infty.$$

The proof of Theorem 3.1.2 is based on two types of random constructions. One of them can be used to obtain a random Cantor set of dimension at most d almost surely; and the other to obtain a random compact set of dimension at least d almost surely.

We will also prove the following generalization of Theorem 3.1.2.

Theorem 3.1.5. *Let $D \subset \mathbb{R}^n$ be a Borel set and let $f : D \to \mathbb{R}^m$ be Borel measurable. Then for every $0 \leq d \leq 1$ there exists a Borel set $A \subset D$ such that $\dim A = d \cdot \dim D$ and $\dim f(A) \leq d \cdot \dim f(D)$.*

These results are based on [6].

2.3 Borel maps and Hausdorff measures

In possession of Theorem 3.1.1, another question arises. Can it happen that $\mathcal{H}^s$ and $2 \cdot \mathcal{H}^s$ are Borel isomorphic? That is, does there exist a Borel bijection f such that

$2 \cdot \mathcal{H}^s(B) = \mathcal{H}^s(f(B))$ for every Borel set B? Of course, if we consider $\mathbb{R}$ (and $\mathbb{R} \to \mathbb{R}$ Borel bijections), then the answer is positive: similarities of ratio $2^{1/s}$ realize the isomorphism. However, if we consider the unit interval $[0,1]$ only, then the question is highly non-trivial. We answer this question by proving the following quantitative sharp version of Theorem 3.1.2.

Theorem 4.5.6. *Let $f : [0,1] \to [0,1]$ be Borel measurable. Then for every $0 < s < 1$ there exists a compact set $C \subset [0,1]$ such that $\mathcal{H}^s(C) = 1$ and $\mathcal{H}^s(f(C)) \leq 1$.*

Corollary 4.5.8. *Let $0 < s < 1$ and $c > 0$, $c \neq 1$ be fixed. Then $\mathcal{H}^s$ and $c \cdot \mathcal{H}^s$ are not Borel isomorphic on $[0,1]$. That is, there does not exist a Borel bijection $f : [0,1] \to [0,1]$ such that $\mathcal{H}^s(B) = c \cdot \mathcal{H}^s(f(B))$ for every Borel set $B \subset [0,1]$.*

The proof of Theorem 4.5.6 is similar to the proof of Theorem 3.1.2. That is, we use two types of random constructions. However, to prove that the random set produced by one of the constructions is sufficiently large, requires much more effort than in the analogous proof of Theorem 3.1.2. This is the reason why we discuss Theorem 3.1.2 separately, despite the fact that Theorem 4.5.6 is a stronger statement.

Theorem 4.5.6 has interesting consequences. Among others, we prove the following two theorems.

Theorem 4.5.17. *Let $f : D \to \mathbb{R}^n$ be a Borel mapping on a Borel set $D \subset \mathbb{R}^n$, and let $0 < s < n$ be fixed. If f does not increase the $\mathcal{H}^s$ measure of any sets, then it does not increase Lebesgue measure either. That is, if*

$$\mathcal{H}^s(A) \geq \mathcal{H}^s(f(A))$$

for every Borel set $A \subset D$, then

$$\lambda(A) \geq \lambda(f(A))$$

for every Borel set $A \subset D$.

Theorem 4.5.21. *Let $n \geq 1$ and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Borel bijection. If f preserves the s-dimensional Hausdorff measure for some $0 < s < n$, then it also preserves the Lebesgue measure. That is, if*

$$\mathcal{H}^s(f^{-1}(B)) = \mathcal{H}^s(B)$$

for every Borel set $B \subset \mathbb{R}^n$, then

$$\lambda(f^{-1}(B)) = \lambda(B)$$

for every measurable set $B \subset \mathbb{R}^n$.

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