

HAUSDORFF DIMENSION OF METRIC SPACES AND LIPSCHITZ MAPS ONTO CUBES

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ABSTRACT. We prove that a compact metric space (or more generally an analytic subset of a complete separable metric space) of Hausdorff dimension bigger than k can be always mapped onto a k -dimensional cube by a Lipschitz map. We also show that this does not hold for arbitrary separable metric spaces.

As an application we essentially answer a question of Urbański by showing that the transfinite Hausdorff dimension (introduced by him) of an analytic subset A of a complete separable metric space is $\lfloor \dim_{\mathbb{H}} A \rfloor$ if $\dim_{\mathbb{H}} A$ is finite but not an integer, $\dim_{\mathbb{H}} A$ or $\dim_{\mathbb{H}} A - 1$ if $\dim_{\mathbb{H}} A$ is an integer and at least ω_0 if $\dim_{\mathbb{H}} A = \infty$.

1. INTRODUCTION

Which compact metric space X can be mapped onto a k -dimensional cube by a Lipschitz map? Since Lipschitz maps can increase the k -dimensional Hausdorff measure by at most a constant multiple, a necessary condition is $\mathcal{H}^k(X) > 0$, where \mathcal{H}^k denotes the k -dimensional Hausdorff measure. Is this condition sufficient? In 1932 Kolmogorov [10] posed a conjecture about minimal measure that would imply an affirmative answer at least for $k = 1$ and $X \subset \mathbb{R}^n$. However, Vitushkin, Ivanov and Melnikov [19] (see also [9] for a less concise proof) constructed a compact subset of the plane with positive 1-dimensional Hausdorff measure that cannot be mapped onto a segment by a Lipschitz map. Konyagin found in the 90's a simpler construction of an abstract compact metric space with the same property but he has not published it.

By proving the following result we show that the condition $\dim_{\mathbb{H}} X > k$, which is just a bit stronger than the necessary condition $\mathcal{H}^k(X) > 0$, is already sufficient.

Theorem 1.1. *If (X, d) is a compact metric space with Hausdorff dimension larger than a positive integer k , then X can be mapped onto a k -dimensional cube by a Lipschitz map.*

In fact we shall prove a more general result (Theorem 2.5) by allowing not only compact metric spaces but also Borel, or even analytic subsets of complete separable metric spaces. But some assumption about the metric space is needed: we show (Theorem 3.5) that there exist separable metric spaces with arbitrarily large Hausdorff dimension that cannot be mapped onto a segment by a uniformly continuous function. Surprisingly, our construction depends on a set theoretical hypothesis that is independent of the standard ZFC axioms: if less than continuum many sets of first category cannot cover the real line then we can give an example in \mathbb{R}^n , otherwise our example is a separable metric space of cardinality less than continuum.

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Recall that an *ultrametric space* is a metric space in which the triangle inequality is replaced with the stronger inequality $d(x, y) \leq \max(d(x, z), d(y, z))$. For compact ultrametric spaces we get the following simple answer to the first question of the introduction.

Theorem 1.2. *A compact ultrametric space can be mapped onto a k -dimensional cube by a Lipschitz map if and only if it has positive k -dimensional Hausdorff measure.*

In fact, we prove (Corollary 2.2) the above result for the following more general (see Lemma 2.3) class of metric spaces.

Definition 1.3. A metric space (X, d) is called *monotone* if there exists a linear order $<$ and a constant C such that

$$(1) \quad \text{diam}([a, b]) \leq C \cdot d(a, b) \quad (\forall a, b \in X),$$

where $[a, b]$ denotes the closed interval $\{x \in X : a \leq x \leq b\}$.

If this holds for a given C then we can also say that the metric space is *C -monotone*.

This notion has been introduced recently by the third author in [21] and it was studied by him and Nekvinda [16, 17]. Our results about monotone spaces are closely related to some of the results of the mentioned papers but for the sake of completeness we present here brief self-contained proofs. Some further closely related results will be published in [20] by the third author.

In section 4 we use our results to discuss the *Urbański conjecture*: In his paper [18] Urbański defined the *transfinite Hausdorff dimension* $\text{tHD}(X)$ of a separable metric space X as the largest possible topological dimension of a Lipschitz image of a subset of X (cf. (2)). He proved that if the Hausdorff dimension of X is finite then it is an upper bound for the transfinite Hausdorff dimension of X (cf. 4.1) and conjectured that, roughly speaking, it is close to a lower bound (cf. 4.2). We show that the conjecture is correct for analytic subsets of complete separable metric spaces, but consistently fails in general.

2. NICE LARGE METRIC SPACES CAN BE MAPPED ONTO CUBES

First we prove the following result about mapping monotone metric spaces onto an interval by Hölder function.

Theorem 2.1. *If (X, d) is a compact monotone metric space with positive s -dimensional Hausdorff measure (for some $s > 0$) then X can be mapped onto a non-degenerate interval by an s -Hölder function.*

Proof. By Frostman lemma we can choose a nonzero finite Borel measure μ on X so that $\mu(E) \leq (\text{diam}(E))^s$ for any $E \subset X$. Since X is a monotone metric space there exists a linear order $<$ and a constant C such that (1) of Definition 1.3 holds. It is easy to show (see also in [17]) that any open interval $(a, b) = \{x \in X : a < x < b\}$ is open, so any interval of X is Borel. For $x \in X$ let $g(x) = \mu((-\infty, x))$, where $(-\infty, x) = \{y \in X : y < x\}$. Then g is s -Hölder since for any $a, b \in X$, $a < b$ we have

$$0 \leq g(b) - g(a) = \mu([a, b]) \leq \text{diam}([a, b])^s \leq (C \cdot d(a, b))^s.$$

Thus g is continuous, so $g(X) \subset \mathbb{R}$ is compact. Since μ is not the zero measure and X is separable, $g(X)$ is not a singleton.

Therefore all we need to prove is that there is no $u, v \in g(X)$ with $u < v$ and $(u, v) \cap g(X) = \emptyset$. Suppose there exist such u and v . Since X is a compact metric space, it is separable, let S be a countable dense subset of X . Let $S_1 = \{s \in S : g(s) \leq u\}$ and $S_2 = \{t \in S : g(t) \geq v\}$. Since $(u, v) \cap g(X) = \emptyset$ we

must have $S = S_1 \cup S_2$. Since S_1 and S_2 are countable and $g(x) = \mu((-\infty, x))$ we get that $\mu(\bigcup_{s \in S_1} (-\infty, s)) \leq u$ and $\mu(\bigcap_{t \in S_2} (-\infty, t)) \geq v$. Hence letting $D = \bigcap_{t \in S_2} (-\infty, t) \setminus \bigcup_{s \in S_1} (-\infty, s)$ we get $\mu(D) \geq v - u > 0$. On the other hand, the set D must have at most two points since if $x, y, z \in D$ and $x < y < z$ then (x, z) would be a nonempty open set not containing any point of the dense set $S = S_1 \cup S_2$. Since $\mu(E) \leq (\text{diam}(E))^s$ for any $E \subset X$, the singletons must have zero μ measure, so we get $\mu(D) = 0$ contradicting the previously obtained $\mu(D) > 0$. \square

Corollary 2.2. *Let X be a compact monotone metric space and let k be a positive integer. Then X can be mapped onto the k -dimensional cube $[0, 1]^k$ by a Lipschitz map if and only if X has positive k -dimensional Hausdorff measure.*

Proof. It is clear that $\mathcal{H}^k(X) > 0$ is a necessary condition.

To prove that it is sufficient note that by the previous theorem there exists a k -Hölder map $g : X \rightarrow \mathbb{R}$ so that $g(X) = [0, 1]$. It is well-known that there exists a $\frac{1}{k}$ -Hölder Peano curve $h : [0, 1] \rightarrow [0, 1]^k$, in fact, one can easily check that any classical construction is $\frac{1}{k}$ -Hölder. Then the composition $h \circ g$ is a Lipschitz map that maps X onto $[0, 1]^k$. \square

Nekvinda and Zindulka [17] proved that an ultrametric space is always a monotone metric space. For compact ultrametric spaces even the following stronger result can be proved fairly easily.

Lemma 2.3. *Any compact ultrametric space (X, d) is 1-monotone.*

Proof. The following tree structure is well-known but as it can be obtained quickly we give a self-contained proof.

Let $D = \text{diam } X$. If $D = 0$ then let $k = 1$ and $X_1 = X$. Otherwise, since d is an ultrametric, the relation $d(x, y) < D$ is an equivalence relation. The equivalence classes are open, X is compact, so there are only finitely many equivalence classes: X_1, \dots, X_k . Thus X_1, \dots, X_k are closed and compact as well. Since X is compact, there exist two points with distance D , so $k \geq 2$, unless $D = 0$. Note also that if a and b are from distinct equivalence classes then $d(a, b) = D$.

Since each equivalence class is a compact ultrametric space we can do the same for each of them. This way we get a tree of clopen sets $X_{i_1 \dots i_m}$ with the property that for a fixed m these sets give a partition of X and if $a \in X_{i_1 \dots i_m j}$, $b \in X_{i_1 \dots i_m j'}$ and $j \neq j'$ then $d(a, b) = \text{diam}(X_{i_1 \dots i_m})$. The first property implies that for any $x \in X$ there exists a unique sequence i_1, i_2, \dots , so that $x \in X_{i_1 \dots i_m}$ for every m . On the other hand, $\lim_{n \rightarrow \infty} \text{diam}(X_{i_1 \dots i_m}) = 0$, since otherwise picking one point from each $X_{i_1 \dots i_m} \setminus X_{i_1 \dots i_{m+1}}$ we would get an infinite discrete subspace. Therefore $\{x\} = \bigcap_{m=1}^{\infty} X_{i_1 \dots i_m}$ and this representation of x is well defined, and then so is the function $J(x) = (i_1, i_2, \dots)$.

We define the order $<$ on X by saying that $x < y$ if $J(x)$ is smaller than $J(y)$ in the lexicographical order. We need to show that for any $a, b \in X$ we have $\text{diam}([a, b]) = d(a, b)$. Let (i_1, \dots, i_m) be the longest common initial segment of a and b . Then, as we saw above, $d(a, b) = \text{diam}(X_{i_1 \dots i_m})$. On the other hand $[a, b] \subset X_{i_1 \dots i_m}$, so we get $d(a, b) \leq \text{diam}([a, b]) \leq \text{diam}(X_{i_1 \dots i_m}) = d(a, b)$, which completes the proof. \square

The following result is a weaker version of [14, Theorem 1.4].

Theorem 2.4 (Mendel and Naor [14]). *For every compact metric space (X, d) and $\varepsilon > 0$ there exists a closed subset $Y \subset X$ such that $\dim_{\text{H}} Y \geq (1 - \varepsilon) \dim_{\text{H}} X$ and (Y, d) is bi-Lipschitz equivalent to an ultrametric space.*

Now we are ready to prove the main result of the paper.

Theorem 2.5. *Let A be an analytic subset of a separable complete metric space (X, d) , and let k be a positive integer. If $\dim_{\mathbb{H}} A > k$ then A can be mapped onto the k -dimensional cube $[0, 1]^k$ by a Lipschitz map.*

Proof. Let $s \in (k, \dim_{\mathbb{H}} A)$. By the theorem of Howroyd [7], A has a compact subset C with finite and positive s -dimensional Hausdorff measure. By the above Mendel–Naor theorem, C has a subset E with $\dim_{\mathbb{H}} E > k$ that is bi-Lipschitz equivalent to an ultrametric space. Applying Howroyd’s theorem again we get a compact subset B of E with positive and finite k -dimensional Hausdorff measure. Clearly, B is also bi-Lipschitz equivalent to a compact ultrametric space Y . By Lemma 2.3 and Corollary 2.2, Y can be mapped onto $[0, 1]^k$ by a Lipschitz map. Since $B \subset A$ is bi-Lipschitz equivalent to Y , this means that B can be also mapped onto $[0, 1]^k$ by a Lipschitz map. Since real valued Lipschitz functions can be always extended as Lipschitz functions, by extending the coordinate functions we get a Lipschitz function that maps A onto $[0, 1]^k$. \square

Remark 2.6. As we saw in the introduction, in the above theorem the condition $\dim_{\mathbb{H}} A > k$ cannot be replaced by the condition that A has positive k -dimensional Hausdorff measure, not even if A is a compact subset of \mathbb{R}^n . However, for $k = n$ this stronger statement might be also true. It is a well-known long standing conjecture of Laczkovich [11] that a measurable set with positive Lebesgue measure in \mathbb{R}^n can be always mapped onto an n -dimensional cube by a Lipschitz map. This is known to be true for $n \leq 2$ (see [1] and [12]), and there is recent (still unpublished) progress for $n \geq 3$ due to Csörnyei and Jones.

Remark 2.7. The following argument shows that if (X, d) is the Euclidean space \mathbb{R}^n , or more generally if it is a complete doubling metric space then we can also prove Theorem 2.5 without using the theorem of Mendel and Naor.

First, let A be an analytic set in \mathbb{R}^n with $\dim_{\mathbb{H}} A > k$. Let $C \subset A$ be compact with $\dim_{\mathbb{H}} C > k$. Choose S to be a self-similar set in \mathbb{R}^n with the strong separation condition (which means that S is the disjoint union of sets similar to S) with $\dim_{\mathbb{H}} S > \max(n - (\dim_{\mathbb{H}} C - k), (n + 1)/2)$. By a theorem of Mattila [13, Theorem 13.11] there exists an isometry φ so that $\dim_{\mathbb{H}} C \cap \varphi(S) > k$. It is easy to check that a self-similar set with the strong separation condition is bi-Lipschitz equivalent to an ultrametric space, so $C \cap \varphi(S)$ is bi-Lipschitz equivalent to a compact ultrametric space X . Since ultrametric spaces are monotone this implies by Corollary 2.2 that $C \cap \varphi(S)$ can be mapped onto $[0, 1]^k$ by a Lipschitz function. By extending the Lipschitz function onto A we get a Lipschitz function that maps A onto $[0, 1]^n$.

By Theorem 2.1 we also get that A can be mapped onto $[0, 1]$ by an s -Hölder function for some $s > k$. By the Assouad embedding theorem [2] for any doubling metric space (X, d) and $\varepsilon > 0$ the metric space $(X, d^{1-\varepsilon})$ admits a bi-Lipschitz embedding into a Euclidean space. Combining these results we get that if B is an analytic subset of a complete doubling metric space (X, d) and $\dim_{\mathbb{H}}(B) > k$ then B can be mapped onto $[0, 1]$ by a k -Hölder function. Then, composing this map with a $\frac{1}{k}$ -Hölder Peano curve as in the proof of Corollary 2.2, we get a Lipschitz map from B onto $[0, 1]^k$.

3. LARGE METRIC SPACES THAT CANNOT BE MAPPED ONTO A SEGMENT

In this section we show that in Theorem 2.5 we need some assumption about A , the theorem would not remain true if we allow an arbitrary separable metric space A .

The following result is probably known but for completeness we present a proof.

Theorem 3.1. *If Continuum Hypothesis holds then for any n there exists a set $A \subset \mathbb{R}^n$ such that for any continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ the set $f(A)$ does not contain any interval and A is not a Lebesgue null set.*

In fact, instead of the CH, it is enough to assume the following hypothesis, which clearly follows from CH.

(\star) *In \mathbb{R}^n less than continuum many closed set of measure zero and a set of measure zero cannot cover \mathbb{R}^n .*

Proof. Let $\{f_\alpha : \alpha < \mathfrak{c}\}$ and $\{N_\alpha : \alpha < \mathfrak{c}\}$ be enumerations of the collection of $\mathbb{R}^n \rightarrow \mathbb{R}$ continuous functions and the collection of Lebesgue null Borel subsets of \mathbb{R}^n , respectively.

By transfinite induction for every $\alpha < \mathfrak{c}$ we construct points $x_\alpha \in \mathbb{R}^n$ and $y_\alpha \in (0, 1)$ such that

- (i) $x_\alpha \notin N_\alpha$,
- (ii) $x_\alpha \notin \bigcup_{\beta < \alpha} f_\beta^{-1}(\{y_\beta\})$,
- (iii) $y_\alpha \notin f_\alpha(\{x_\beta : \beta \leq \alpha\})$ and
- (iv) $f_\alpha^{-1}(\{y_\alpha\})$ has Lebesgue measure zero.

In the α -th step we suppose that (i)–(iv) hold for smaller indices. Since for each $\beta < \alpha$ the set $f_\beta^{-1}(\{y_\beta\})$ is a closed set of measure zero, (\star) implies that we can choose x_α so that (i) and (ii) hold. We can choose $y_\alpha \in (0, 1)$ so that (iii) and (iv) hold since more than countably many of the pairwise disjoint closed sets $f_\alpha^{-1}(\{t\})$ ($t \in (0, 1)$) cannot have positive measure.

Let $A = \{x_\alpha : \alpha < \mathfrak{c}\}$. This set cannot have zero measure by (i). If there exists a continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ so that $f(A)$ contains an interval then the image of a linear transformation contains $(0, 1)$, so there exists an $\alpha < \mathfrak{c}$ for which $f_\alpha(A) \supset (0, 1)$. But this is impossible since $f_\alpha(x_\beta) \neq y_\alpha$ for any $\beta \leq \alpha$ by (iii) and for any $\beta > \alpha$ by (ii), so $y_\alpha \notin f_\alpha(A)$. \square

Since real valued uniformly continuous functions can be always extended we get the following.

Corollary 3.2. *If Continuum Hypothesis (or (\star) of Theorem 3.1) holds then for any n there exists a set $A \subset \mathbb{R}^n$ such that no subset of A can be mapped onto a segment by a uniformly continuous map and A is not a Lebesgue null set, so it has Hausdorff dimension n .*

Recall that $\text{cov } \mathcal{M}$ denotes the least cardinality of a collection of meager sets that can cover \mathbb{R}^n . Note that this definition does not depend on n (see e.g. [4]).

Remark 3.3. By a well-known theorem of Bartoszyński and Shelah [4, Theorem 2.6.14] $\text{cov } \mathcal{M}$ is also the least cardinality of a collection of closed sets of measure zero so that the union is not a set of measure zero. Hence $\text{cov } \mathcal{M} = \mathfrak{c}$ implies (\star) of Theorem 3.1, therefore in Theorem 3.1 and Corollary 3.2 it is enough to assume $\text{cov } \mathcal{M} = \mathfrak{c}$.

Our next goal is to handle the $\text{cov } \mathcal{M} < \mathfrak{c}$ case by constructing a separable metric space (X, d) of cardinality $\text{cov } \mathcal{M}$ with arbitrarily large Hausdorff dimension. Let \mathcal{H}^φ denote the Hausdorff measure with strictly increasing bijective gauge function $\varphi : [0, \infty) \rightarrow [0, \infty)$. A metric space X is of *strong measure zero* if $\mathcal{H}^\varphi(X) = 0$ for any gauge function φ .

We construct our example in ω^ω . For $x \neq y$ in ω^ω denote by $|x \wedge y|$ the length of the common initial segment of x and y . Note that, for any decreasing function $g : \{1, 2, \dots\} \rightarrow (0, \infty)$ with $\lim_n g(n) = 0$, by letting $d_g(x, y) = g(|x \wedge y| + 1)$ we get a separable metric space (ω^ω, d_g) . One of the classical metrics on ω^ω is d_{g_0} for $g_0(m) = 1/m$.

Theorem 3.4. *For any gauge function φ there exists a separable metric space (X, d) of cardinality $\text{cov } \mathcal{M}$ with $\mathcal{H}^\varphi((X, d)) > 0$.*

Proof. Fremlin and Miller [15, Theorem 5] proved that $\text{cov } \mathcal{M}$ is also the least cardinality of a subset of (ω^ω, d_{g_0}) that is not a strong measure zero subspace. Therefore there exists $H \subset \omega^\omega$ of cardinality $\text{cov } \mathcal{M}$ such that (H, d_{g_0}) is not a strong measure zero metric space, so there exists a gauge function φ_0 such that $\mathcal{H}^{\varphi_0}(H, d_{g_0}) > 0$. Set $g = \varphi^{-1} \circ \varphi_0 \circ g_0$. Then $\varphi_0 \circ g_0 = \varphi \circ g$ and so $\mathcal{H}^\varphi(H, d_g) = \mathcal{H}^{\varphi_0}(H, d_{g_0}) > 0$. Therefore (H, d_g) is a separable metric space of cardinality $\text{cov } \mathcal{M}$ with $\mathcal{H}^\varphi((H, d_g)) > 0$. \square

Therefore, combining Corollary 3.2, Remark 3.3 and Theorem 3.4 we get the main result of this section.

Theorem 3.5. *There exist separable metric spaces with arbitrarily large Hausdorff dimension that cannot be mapped onto a segment by a uniformly continuous function.*

Remark 3.6. As we saw in Corollary 3.2, if $\text{cov } \mathcal{M} = \mathfrak{c}$ then this metric space can be chosen as a subspace of a Euclidean space. However, some set theoretical assumption is needed, Corollary 3.2 cannot be proved in ZFC: Corazza [5] constructed a model in which every positive Hausdorff dimensional subset of a Euclidean space can be mapped onto $[0, 1]$ by a uniformly continuous function.

We do not know whether the existence of finite Hausdorff dimensional examples can be proved in ZFC. We do not know either whether Corollary 3.2 can be proved in ZFC for Lipschitz maps.

4. TRANSFINITE HAUSDORFF DIMENSION

Urbański [18] introduced the transfinite Hausdorff dimension (tHD) of a metric space X in the following way:

$$(2) \quad \text{tHD}(X) = \sup\{\text{ind}(f(Y)) : Y \subset X, f : Y \rightarrow Z \text{ Lipschitz, } Z \text{ a metric space}\},$$

where ind denotes the transfinite small inductive topological dimension (see e.g. in [6]). He showed the following connection between Hausdorff dimension and transfinite Hausdorff dimension.

Theorem 4.1 (Urbański [18, Theorem 2.8]). *If X is a metric space with finite Hausdorff dimension then $\text{tHD}(X) \leq \lfloor \dim_{\text{H}} X \rfloor$, where $\lfloor \cdot \rfloor$ denotes the floor function.*

After noticing that equality is not always true (if C is a Cantor set with zero Lebesgue measure but Hausdorff dimension 1 then $\text{tHD}(C) = 0$) Urbański stated the following conjecture.

Conjecture 4.2 (Urbański [18, Conjecture 6.1]). *If X is a metric space with finite Hausdorff dimension then $\text{tHD}(X) \geq \lfloor \dim_{\text{H}} X \rfloor - 1$. Consequently $\text{tHD}(X) \in \{\lfloor \dim_{\text{H}} X \rfloor - 1, \lfloor \dim_{\text{H}} X \rfloor\}$.*

It is not hard to see that one cannot prove this conjecture without some additional assumptions: It is well-known (see e.g. in [4]) that for any n the existence of a set $X \subset \mathbb{R}^n$ with positive Lebesgue outer measure and cardinality less than continuum is consistent with ZFC. Then $\dim_{\text{H}} X = n$ but $\text{tHD}(X) = 0$ since any image of any subset of X has cardinality less than continuum and it is easy to check that any set of cardinality less than continuum must have zero topological dimension (since we can easily find balls of arbitrarily small radius with empty boundary around any point).

Theorem 2.5 immediately implies that if we have some reasonable assumption about the metric space than even the following stronger form of Conjecture 4.2 holds.

Theorem 4.3. *If A is an analytic subset of a complete separable metric space then $\text{tHD}(A) \geq \lceil \dim_{\text{H}} A \rceil - 1$, where $\lceil \cdot \rceil$ denotes the ceiling function.*

Combining Theorems 4.1 and 4.3 we get the following.

Corollary 4.4. *Let A be an analytic subset of a complete separable metric space.*

- *If $\dim_{\text{H}} A$ is finite but not an integer then $\text{tHD}(A) = \lfloor \dim_{\text{H}} A \rfloor$,*
- *if $\dim_{\text{H}} A$ is an integer then $\text{tHD}(A)$ is $\dim_{\text{H}} A$ or $\dim_{\text{H}} A - 1$, and*
- *if $\dim_{\text{H}} A = \infty$ then $\text{tHD}(A) \geq \omega_0$.*

Remark 4.5. Corollary 4.4 is sharp in the sense that if $\dim_{\text{H}} A = n \in \mathbb{N}$ then $\text{tHD}(A)$ can be both n and $n - 1$: the trivial $A = \mathbb{R}^n$ is clearly an example for the former one and we claim that any A with zero n -dimensional Hausdorff measure is an example for the latter one. It is well-known (see e.g. [8]) that the topological dimension of a metric space with zero n -dimensional Hausdorff measure is at most $n - 1$. Therefore if A has zero n -dimensional Hausdorff measure then the Lipschitz image of any of its subsets also has zero n -dimensional Hausdorff measure, so its topological dimension is at most $n - 1$, which means that $\text{tHD}(A) \leq n - 1$.

Our final goal is to show (in ZFC) that Theorem 4.3 and Corollary 4.4 do not hold for a general separable metric space. The proof of the following useful observation uses a well-known argument that can be found for example in [3] or [22, 4.2].

Lemma 4.6. *For any metric space X , $\text{tHD}(X) = 0$ if and only if X cannot be mapped onto a segment by a Lipschitz map.*

Proof. It is clear that if $\text{tHD}(X) = 0$ then X cannot be mapped onto a segment by a Lipschitz map. To prove the converse suppose that $Y \subset X$ and $f : Y \rightarrow Z$ is a Lipschitz map onto a metric space (Z, ρ) of positive topological dimension. The latter implies that there is $z_0 \in Z$ and $r > 0$ such that every sphere $\{z \in Z : \rho(z_0, z) = s\}$ of radius $s \leq r$ is nonempty. It follows that the Lipschitz function $g(z) = \min(\rho(z_0, z), r)$ maps Z onto $[0, r]$. Thus a Lipschitz extension of $g \circ f$ maps X onto $[0, r]$. \square

Combining Theorem 3.5 and Lemma 4.6 we get the following.

Theorem 4.7. *There exist separable metric spaces with zero transfinite Hausdorff dimension and arbitrarily large Hausdorff dimension.*

Remark 4.8. We do not know whether in ZFC one can give an example with finite Hausdorff dimension and we cannot disprove the original form of Conjecture 4.2 in ZFC either.

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