Measurable functions are of bounded variation on a set of dimension 1/2

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Abstract

We show that for every Lebesgue measurable function $f : [0,1] \to \mathbb{R}$ there exists a compact set C of Hausdorff dimension $1/2$ such that f is of bounded variation on C, and there exist compact sets C_α of Hausdorff dimension $1-\alpha$ such that f is Hölder-α on C_α $(0 < \alpha < 1)$. These answer questions of M. Elekes, which were open even for continuous functions f. Our proof goes by defining a discrete Hausdorff pre-measure on $\mathbb{Z}$, solving the corresponding discrete problems, and then finding suitable limit theorems.

1 Introduction

It was an unsolved problem for several years whether Hausdorff measures of different dimensions can be Borel isomorphic or not. This problem is attributed to B. Weiss and popularized by D. Preiss (see also [6]). Let $\mathcal{B}$ denote the Borel σ-algebra of $\mathbb{R}$, and $\mathcal{H}^d$ denote the d-dimensional Hausdorff measure; then the exact question reads as follows.

Question 1.1.

(i) Can the measure spaces $(\mathbb{R}, \mathcal{B}, \mathcal{H}^s)$ and $(\mathbb{R}, \mathcal{B}, \mathcal{H}^t)$ be isomorphic if $s \neq t$ $(s, t \in [0,1])$?

(ii) Let $0 < s < t < 1$. Does there exist a Borel bijection $f : \mathbb{R} \to \mathbb{R}$ such that for every Borel set B,
$$0 < \mathcal{H}^s(B) < \infty \iff 0 < \mathcal{H}^t(f(B)) < \infty?$$

The two parts are not equivalent but it is easy to see that the negative answer to (ii) implies the negative answer to (i).

M. Elekes, aiming to solve this problem, raised the following question [4].

Question 1.2. Can we find for every Borel (or continuous, or typical[1] continuous) function $f : [0,1] \to \mathbb{R}$ a Borel set B of positive Hausdorff dimension such that f restricted to B is Hölder continuous of exponent α? $(0 < \alpha \leq 1)$.

How is this question related to the previous? Suppose that we have an answer to Question 1.2 so that for every Borel function f there exists a Borel set B of dimension β, such that f is Hölder-α on B (for some fixed α). As it is well-known, this implies that $f(B)$ has dimension at most β/α. It is easy to see that this would answer (both parts of) Question 1.1 in the negative for those s and t for which $0 < s < \beta < \beta/\alpha < t < 1$ holds.

According to a theorem of P. Humke and M. Laczkovich [5], a typical continuous function $f : [0,1] \to \mathbb{R}$ is not monotone on any set of positive Hausdorff dimension. Since every function of bounded variation is the sum of two monotone functions, this result motivated M. Elekes to raise an analogue of Question 1.2.

Question 1.3. Can we find for every Borel (or continuous, or typical continuous) function $f : [0,1] \to \mathbb{R}$ a Borel set B of positive Hausdorff dimension such that f restricted to B is of bounded variation? Can we even find a set of dimension $1/2$?

This problem is also circulated by D. Preiss, and a similar question was already asked by P. Humke and M. Laczkovich, see also Z. Buczolich [1, 2].

M. Elekes proved in [4] that a typical continuous function $f : [0,1] \to \mathbb{R}$ is not of bounded variation on any set of Hausdorff dimension larger than $1/2$. Regarding Question 1.2 he also gave an upper bound for the possible dimension by showing that for every $0 < \alpha \leq 1$, a typical continuous function is not Hölder-α on any set of dimension larger than $1 - \alpha$. However, no other result was known so far regarding these questions.

Our goal here is to answer Question 1.3 and Question 1.2 in the strongest possible form.

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[1] in the Baire category sense

Theorem 1.4. *Let $f : [0,1] \to \mathbb{R}$ be Lebesgue measurable. Then there exists a compact set $C \subset [0,1]$ of Hausdorff dimension $1/2$ such that $f|_C$ is of bounded variation.*

Theorem 1.5. *Let $f : [0,1] \to \mathbb{R}$ be Lebesgue measurable and $0 < \alpha < 1$. Then there exists a compact set $C \subset [0,1]$ of Hausdorff dimension $1-\alpha$ such that f restricted to C is a Hölder-α function.*

As described above, the second theorem gives some partial results on the isomorphism problem of Hausdorff measures (Question 1.1). However, recently this problem has been solved completely by the present author [6]. The proof does not rely on Theorem 1.5, though the questions and work of M. Elekes were definitely inspiring: In [6] it is proved that for every Borel measurable mapping $f : \mathbb{R} \to \mathbb{R}$ for every $d \in [0,1]$ there exists a compact set C of dimension d such that $f(C)$ is of dimension at most d. The proof is based on a random construction, and it is very different from the proofs of the present article.

We shall prove Theorem 1.4 and 1.5 parallelly. At first we shall define the discrete Hausdorff pre-measure on $\mathbb{Z}$ (the integers) and prove some basic facts about it (§2). Using this, we will be able to formalize and prove the corresponding problems in a discrete setting (Theorem 3.3 and 3.4) in §3. Then in §4 we prove suitable limit theorems which together with Theorem 3.3 and 3.4 yield the proof of Theorem 1.4 and 1.5. There is also a brief overview of the whole proof at the beginning of §3. We mention some open questions and generalizations of Theorem 1.4 and 1.5 in §5.

Theorem 1.4 and 1.5 belong to the family of *restriction theorems*. The setting of a restriction theorem usually is the following. Given some function f from some class X, one tries to find a large set A so that $f|_A$ belongs to some other (nice) class Y. Here largeness usually means that A is infinite, uncountable, perfect, not porous, or A is of positive measure or of second category. It is interesting that for the above questions of M. Elekes, the proper notion of largeness is some specific Hausdorff dimension. We refer the reader to the survey article of J. B. Brown [3] on restriction theorems and to the references therein.

Definition 1.6. We say that the real function f is of bounded variation on the set A if f restricted to A is a function of bounded variation. We say that the real function f is Hölder continuous of exponent α (or briefly Hölder-α) on the set A if $f|_A$ is Hölder-α; that is, there exists a real number $B > 0$ such that for every $x, y \in A$, $|f(x) - f(y)| \leq B|x-y|^\alpha$.

Notation. We denote the set of nonnegative integers by $\mathbb{N}$. We adopt the brief notation $n = \{0, 1, 2, \ldots, n-1\}$ for each $n \in \mathbb{N}$ as usual in set theory. However, n will always denote a positive integer in this paper (unless otherwise stated).

Let $\mathcal{H}^s_\infty(A)$ denote the s-dimensional Hausdorff pre-measure of the set $A \subset \mathbb{R}$; that is,

$$\mathcal{H}^s_\infty(A) = \inf\{\textstyle\sum_{i=1}^\infty (\operatorname{diam} I_i)^s \ : \ A \subset \cup_{i=1}^\infty I_i\}.$$

By dimension we always mean the Hausdorff dimension.

For $x \in \mathbb{R}$, we denote by $\lceil x \rceil$ the smallest integer which is not smaller than x.

If $\emptyset \neq A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$, we denote the total variation of f by $\operatorname{Var} f$.

If $\emptyset \neq A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$, we say that $f \in B$-Hölder$^\alpha$ if

$$\forall\, x, y \in A \ \ |f(x) - f(y)| \leq B|x-y|^\alpha.$$

Remark 1.7. Notice that if $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$ is a given function, then there exists a function $g : \mathbb{R} \to \mathbb{R}$ extending f (that is, $g|_A = f$) such that the total variation of g and f are equal and that f is Hölder-α if and only if g is Hölder-α ($0 < \alpha \leq 1$). (Given f, one can easily define g on the closure of A, and then the linear extension works.) This yields that the following are equivalent for every function $f : [0,1] \to \mathbb{R}$ and $\beta \in [0,1]$:

(i) There exists a set A of dimension at least β so that $f|_A$ is of bounded variation;

(ii) There exists a function $g : [0,1] \to \mathbb{R}$ of bounded variation so that the set $[f = g]$ (that is, $\{x : f(x) = g(x)\}$) has dimension at least β;

and the same equivalence holds for the Hölder-α property. Thus Question 1.3 and 1.2 and Theorem 1.4 and 1.5 could have also been formulated equivalently corresponding to (ii).

2 Discrete Hausdorff measure

We define the discrete Hausdorff pre-measure on the subsets of the integers $\mathbb{Z}$. The covering sets will be intervals $I \subset \mathbb{Z}$, and here by interval we mean a set of finitely many consecutive integers. By $|I|$ we denote the number of elements of I.

Let $d(X, Y)$ denote the usual distance of X and Y ($X, Y \subset \mathbb{R}$). If X or Y is empty, then we define their distance to be ∞.

Definition 2.1. Let $0 < s \leq 1$. The discrete Hausdorff pre-measure of dimension s is the function $\mu^s : \mathcal{P}(\mathbb{Z}) \to [0, \infty]$ defined by

$$\mu^s(A) = \min\{\sum_{I \in \mathcal{I}} |I|^s : \mathcal{I} \text{ is a collection of intervals of } \mathbb{Z} \text{ such that } A \subset \cup\mathcal{I}\}.$$

It is reasonable to call μ^s a pre-measure since it is subadditive:

Lemma 2.2. *Let $0 < s \leq 1$ and $A, B \subset \mathbb{Z}$. Then*

$$\mu^s(A \cup B) \leq \mu^s(A) + \mu^s(B).$$

We also have an inequality the other way around.

Lemma 2.3. *Let $0 < s \leq 1$ and $A, B \subset \mathbb{Z}$. Then*

$$\mu^s(A \cup B) \geq \min\Big(d(A, B)^s,\ \mu^s(A) + \mu^s(B)\Big).$$

Proof. Consider a covering of $A \cup B$ with intervals of integers. If there is an interval of size at least $d(A, B)$ then the inequality clearly holds. Suppose that every interval has a size at most $d(A, B)$. Then each interval can intersect either A or B but not both, so we can divide up the covering into two parts to cover A and to cover B, which corresponds to the case $\mu^s(A \cup B) \geq \mu^s(A) + \mu^s(B)$. $\square$

The following statement connects μ^s to the (real) Hausdorff pre-measure $\mathcal{H}^s_\infty$.

Lemma 2.4. *Let $0 < s \leq 1$. For a set $A \subset \mathbb{Z}$, let us define*

$$A^* \stackrel{def}{=} \cup_i \big\{[i,\ i + \tfrac{1}{2}] : i \in A\big\}.$$

Then

$$\mathcal{H}^s_\infty(A^*) \leq \mu^s(A) \leq 2^s \cdot \mathcal{H}^s_\infty(A^*).$$

Proof. We may suppose that A is finite (that is, bounded). The left hand side is immediately trivial if we exchange each covering interval $I \subset \mathbb{Z}$ of A to the interval $[\min I, \max I + 1]$.

It is well-known that it is enough to consider only finite coverings of A^* with closed intervals to calculate $\mathcal{H}^s_\infty(A^*)$. Notice that we may suppose that the covering intervals are disjoint since $(a + b)^s \leq a^s + b^s$ for every $a, b \geq 0$. Hence we may also suppose that all the intervals covering A^* are of the form $[n, n + l + \frac{1}{2}]$ (with length $l + \frac{1}{2}$) for some integer n and $l \in \mathbb{N}$. Hence one can cover A with the corresponding intervals $\{n, n + 1, \ldots, n + l\}$ of size $l + 1$. Since $(l + 1)^s \leq 2^s(l + \frac{1}{2})^s$ for every $l \in \mathbb{N}$, we obtain the inequality. $\square$

Definition 2.5. Let $A \subset \mathbb{Z}$. We say that a mapping $\varphi : A \to \mathbb{Z}$ is *non-contractive* if $|\varphi(x) - \varphi(y)| \geq |x - y|$ for every $x, y \in A$.

We shall use the following observation many times in the followings.

Lemma 2.6. *If $\varphi : A \to \mathbb{Z}$ is non-contractive, then $\mu^s(\varphi(A)) \geq \mu^s(A)$.*

The proof is left to the reader.

3 Discrete version

Before we start we would like to motivate the theorems and proofs of this section by giving an informal overview of the proof of Theorem 1.4 and 1.5.

At first let us just consider the case of variations (that is, Theorem 1.4). So let $f : [0, 1] \to \mathbb{R}$ be measurable. Then f is continuous on some compact set of positive measure. Let us now just suppose that f is continuous on the whole interval $[0, 1]$, it will not make much difference. We would like to prove that f possesses the property that there exists a set $C \subset [0, 1]$ of large dimension so that $f|_C$ has finite variation. A key observation is that this property (or at least a quantitative version of this property) goes through uniform convergence. That is, if some (not necessarily continuous) functions f_n converge uniformly to f, and there exist compact sets C_n such that

$$\mathcal{H}^s_\infty(C_n) \geq \varepsilon \quad \text{and} \quad \operatorname{Var} f_n|_{C_n} \leq B$$

for some $\varepsilon > 0$ and finite B, then there exists a compact set C such that

$$\mathcal{H}^s_\infty(C) \geq \varepsilon \quad \text{and} \quad \operatorname{Var} f|_C \leq B.$$

(Note that $\mathcal{H}^s_\infty(C) > 0$ implies that the dimension of C is at least s.) Hence it is enough to show that this (quantitative) property holds for a dense family of functions. We choose the family of those functions g which are piecewise constant on the intervals $[\frac{i}{n}, \frac{i+1}{n})$ $(i = 0, \ldots, n - 1)$. But for simplicity, we will rather deal with the discrete functions $h : n \to \mathbb{R}$ related to g by $h(i) = g(\frac{i}{n})$ $(i = 0, \ldots, n - 1)$.

Now it is not difficult (using Lemma 2.4) to relate the two properties that

(i) there exists a compact set $C \subset [0,1]$ such that $\mathcal{H}^s_\infty(C)$ is large and $\operatorname{Var} g|_C$ is small; and

(ii) there exists a set $A \subset n$ such that $\mu^s(A)$ is large and $\operatorname{Var} h|_A$ is small.

Thus we only need to show that statement (ii) (after properly formulated) holds for all n and all functions $h : n \to \mathbb{R}$, when $s = 1/2$. (Unfortunately, we can show this for any fixed $s < 1/2$ only, but this will be enough to prove Theorem 1.4.) In some sense, the proof will go by induction on n. This statement is what we will formulate precisely and prove in this section.

Regarding Theorem 1.5, an analogous informal proof could be told, but one has to change phrases like "small variation" to "Hölder-α with small constant"; that is, $\operatorname{Var} f|_C \le B$ to $f|_C \in B$-Hölder$^\alpha$.

3.1 Formalizing the discrete problem

In the following definitions n is a positive integer, B is a positive real, and $s, \alpha \in (0,1]$.

Definition 3.1.

$$b(n, B, s) = \min_{f:n\to[0,1]} \max_{A\subset n} \big\{\mu^s(A) : \operatorname{Var} f|_A \le B\big\}.$$

Definition 3.2.

$$c_\alpha(n, B, s) = \min_{f:n\to[0,1]} \max_{A\subset n} \big\{\mu^s(A) : f|_A \in B\text{-Hölder}^\alpha\big\}.$$

Notice that b and c are monotonic increasing in n and in B. Clearly $b(n, B, s) \ge 1$ and $c_\alpha(n, B, s) \ge 1$ for all n, since the μ^s-measure of a single point is 1.

The discrete analogues of Theorems 1.4 and 1.5 are the following.

Theorem 3.3. *For every* $0 < s < 1/2$ *and* $B > 0$,

$$\inf_{n\ge1} \frac{b(n, B, s)}{n^s} > 0.$$

Theorem 3.4. *Given any* $0 < \alpha < 1$, *for every* $0 < s < 1 - \alpha$ *and* $B > 0$,

$$\inf_{n\ge1} \frac{c_\alpha(n, B/n^\alpha, s)}{n^s} > 0.$$

Note that the denominator n^s is present because when we exchange a function $g : [0,1] \to \mathbb{R}$ for the function $h : n \to \mathbb{R}$ (as in the informal overview at the beginning of this section), there is a scaling by a factor of n, and this changes s-dimensional measures by a factor of n^s. Also note that while variation does not change when exchanging g for h, the Hölder-α constant "B" does change by a factor of n^α. This explains the difference between the numerators in the theorems.

3.2 Bounded variation

We start with the proof of Theorem 3.3, because variation is slightly easier to handle than the Hölder property. However, the two proofs are analogous. At first we state and prove two main "induction steps", and what remains, will be just calculations.

Lemma 3.5. *Fix a positive integer* K. *Then for every* $s < 1/2$ *and* $B > 0$

$$b(n, KB + K - 1, s) \ge \min\Big(\big(\tfrac{n}{4K}\big)^s,\ K \cdot b(\lceil \tfrac{n}{2K} \rceil, B, s)\Big)$$

for every $n \in \mathbb{N}$ *large enough (depending only on* K*).*

Proof. Let us choose K intervals of integers $I_1, \dots, I_K$ inside $n = \{0, 1, \dots, n-1\}$ of size $\lceil \frac{n}{2K} \rceil$ such that the distance of every two of them is at least $\frac{n}{4K}$; clearly this can be done if n is sufficiently large (depending on K).

Fix any function $f : n \to [0,1]$. We have to find a set $A \subset n$ of large μ^s-measure such that $\operatorname{Var} f|_A \le KB + K - 1$. For each interval I_j, consider the function $f|_{I_j}$. Since $|I_j| = \lceil \frac{n}{2K} \rceil$, by the definition of $b(\lceil \frac{n}{2K} \rceil, B, s)$ we can find a set $A_j \subset I_j$ such that

$$\operatorname{Var} f|_{A_j} \le B \qquad \text{and} \qquad \mu^s(A_j) \ge b(\lceil \tfrac{n}{2K} \rceil, B, s).$$

Put $A = \cup_{j=1}^K A_j$. Applying Lemma 2.3 inductively to the sets A_j we get

$$\mu^s(A) \ge \min\Big(\big(\tfrac{n}{4K}\big)^s,\ K \cdot b(\lceil \tfrac{n}{2K} \rceil, B, s)\Big). \tag{1}$$

Since $\operatorname{Var} f|_A \le \sum_{j=1}^K \operatorname{Var} f|_{A_j} + (K-1) \le KB + K - 1$, (1) instantly gives Lemma 3.5. ☐

Lemma 3.6. *For each positive integer* L,

$$b(n, B, s) \ge b(\lceil \tfrac{n}{L} \rceil, BL, s).$$

Proof. Fix any function $f : n \to [0,1]$. We have to find a set $A \subset n$ such that $\operatorname{Var} f|_A \le B$ and that $\mu^s(A) \ge b(\lceil \frac{n}{L} \rceil, BL, s)$. For each $i \in L$ let

$$S_i = \{x \in n : \tfrac{i}{L} \le f(x) \le \tfrac{i+1}{L}\}.$$

There exists an $i \in L$ such that $|S_i| \ge \lceil \frac{n}{L} \rceil$; let S be a subset of this S_i of size exactly $|S| = \lceil \frac{n}{L} \rceil$. Let $\varphi : |S| \to S$ be the enumeration of S; that is, φ is the monotonic increasing bijection from $|S|$ to S. Thus φ is a non-contractive mapping. Define $g : |S| \to [0,1]$ by setting

$$g(x) = L \cdot \big(f(\varphi(x)) - \tfrac{i}{L}\big). \tag{2}$$

By the definition of $b(|S|, BL, s)$, there exists a set $T \subset |S|$ such that

$$\mu^s(T) \ge b(|S|, BL, s) \quad \text{and} \quad \operatorname{Var} g|_T \le BL.$$

Using Lemma 2.6 and (2),

$$\mu^s(\varphi(T)) \ge \mu^s(T) \ge b(|S|, BL, s) \quad \text{and} \quad \operatorname{Var} f|_{\varphi(T)} = \tfrac{1}{L} \cdot \operatorname{Var} g|_T \le B.$$

Thus A can be chosen as $\varphi(T)$, which proves this Lemma. □

Proof of Theorem 3.3. We consider $s < 1/2$ to be fixed. Let K be a sufficiently large positive integer, in fact, let $K > 2^{2s} K^{2s}$ hold. At first we will prove the Theorem for $B = 2K - 1$.

Let us apply Lemma 3.5 with $B = 1$. We obtain an $N \in \mathbb{N}$ such that for all $n \ge N$,

$$b(n, 2K-1, s) \ge \min\big(\big(\tfrac{n}{4K}\big)^s, K \cdot b(\lceil \tfrac{n}{2K} \rceil, 1, s)\big).$$

Now apply Lemma 3.6 for the right hand side with $L = 2K - 1$. We obtain

$$b(n, 2K-1, s) \ge \min\Big(\big(\tfrac{n}{4K}\big)^s, K \cdot b(\lceil \frac{\lceil \frac{n}{2K} \rceil}{2K-1} \rceil, 2K-1, s)\Big) \qquad (n \ge N).$$

Since b is monotonic increasing in its first coordinate,

$$b(n, 2K-1, s) \ge \min\big(\big(\tfrac{n}{4K}\big)^s, K \cdot b(\lceil \tfrac{n}{2K(2K-1)} \rceil, 2K-1, s)\big) \qquad (n \ge N). \tag{3}$$

Fix an arbitrary positive integer n_0, and define the series

$$n_{i+1} = \lceil \tfrac{n_i}{2K(2K-1)} \rceil \qquad (i \in \mathbb{N}). \tag{4}$$

Let j be the smallest nonnegative integer for which either

$$b(n_j, 2K-1, s) \ge \big(\tfrac{n_j}{4K}\big)^s$$

or $n_j \le N$ holds. Thus from (3) we obtain that

$$b(n_i, 2K-1, s) \ge K \cdot b(n_{i+1}, 2K-1, s) \qquad (0 \le i < j), \tag{5}$$

and

$$b(n_j, 2K-1, s) \ge \big(\tfrac{n_j}{4KN}\big)^s, \tag{6}$$

since if $n_j \le N$, then the right hand side is smaller than 1, which is a trivial lower bound. Thus from (5) and (6) we get

$$b(n_0, 2K-1, s) \ge K^j \cdot \big(\tfrac{n_j}{4KN}\big)^s.$$

Using (4) we obtain the lower bound

$$\begin{aligned} b(n_0, 2K-1, s) &\ge K^j \cdot \big(\tfrac{n_j}{4KN}\big)^s \ge K^j \cdot \big(\tfrac{1}{4KN}\big)^s n_0^s \big(\tfrac{1}{2K(2K-1)}\big)^{js} \\ &= \big(\tfrac{1}{4KN}\big)^s n_0^s \big(\tfrac{K}{2^s K^s (2K-1)^s}\big)^j \ge \big(\tfrac{1}{4KN}\big)^s n_0^s \end{aligned} \tag{7}$$

provided that $\frac{K}{2^s K^s (2K-1)^s} > 1$, which clearly holds since K was chosen so that $K > 2^{2s} K^{2s}$ holds. Since n_0 was arbitrary, from (7) we immediately obtain that

$$\inf_{n \ge 1} \frac{b(n, 2K-1, s)}{n^s} \ge \big(\tfrac{1}{4KN}\big)^s > 0. \tag{8}$$

Now let $B > 0$ be arbitrary. Let $L \in \mathbb{N}$ be so large that $BL \ge 2K - 1$ holds. Using Lemma 3.6, the fact that b is monotonic increasing in its second coordinate, and then (8),

$$\inf_{n \ge 1} \frac{b(n, B, s)}{n^s} \ge \inf_{n \ge 1} \frac{b(\lceil \frac{n}{L} \rceil, BL, s)}{n^s} \ge \inf_{n \ge 1} \frac{b(\lceil \frac{n}{L} \rceil, 2K-1, s)}{n^s} \ge \inf_{n' \ge 1} \frac{b(n', 2K-1, s)}{(n'L)^s} > 0.$$

□

3.3 Hölder-α

To prove Theorem 3.4, we also start with two "induction steps" (the analogues of Lemma 3.5 and 3.6).

Lemma 3.7. *Fix a positive integer K. There exists an $N \in \mathbb{N}$ such that for every $0 < \alpha \leq 1$, $0 < s < 1$ and $n \geq N$,*

$$c_\alpha(n, B, s) \geq \min\left(\left(\tfrac{n}{4K}\right)^s, K \cdot c_\alpha(\lceil \tfrac{n}{2K} \rceil, B, s)\right)$$

if $B \geq \frac{4K}{n^\alpha}$.

Proof. Let us choose K intervals of integers $I_1, \dots, I_K$ inside $n = \{0, 1, \dots, n-1\}$ of size $\lceil \frac{n}{2K} \rceil$ such that the distance of every two of them is at least $\frac{n}{4K}$; clearly this can be done if n is sufficiently large (depending on K).

Fix any function $f : n \to [0, 1]$. We have to find a set $A \subset n$ of large μ^s-measure such that $f|_A \in B$-Hölder$^\alpha$. For each interval I_j, consider the function $f|_{I_j}$. Since $|I_j| = \lceil \frac{n}{2K} \rceil$, by the definition of $c_\alpha(\lceil \frac{n}{2K} \rceil, B, s)$ we can find a set $A_j \subset I_j$ such that

$$f|_{A_j} \in B\text{-Hölder}^\alpha \quad \text{and} \quad \mu^s(A_j) \geq c_\alpha(\lceil \tfrac{n}{2K} \rceil, B, s).$$

Put $A = \cup_{j=1}^K A_j$. Applying Lemma 2.3 inductively to the sets A_j we get

$$\mu^s(A) \geq \min\left(\left(\tfrac{n}{4K}\right)^s, K \cdot c_\alpha(\lceil \tfrac{n}{2K} \rceil, B, s)\right). \tag{9}$$

It is easy to see that $f|_A \in B$-Hölder$^\alpha$ if $1 \leq B \cdot \left(\frac{n}{4K}\right)^\alpha$, which holds since $\frac{4K}{n^\alpha} \leq B$. Thus (9) gives the proof. ☐

Lemma 3.8. *For each positive integer L,*

$$c_\alpha(n, B, s) \geq c_\alpha(\lceil \tfrac{n}{L} \rceil, BL, s).$$

Proof. Fix any function $f : n \to [0, 1]$. We have to find a set $A \subset n$ such that $f|_A \in B$-Hölder$^\alpha$ and that $\mu^s(A) \geq c_\alpha(\lceil \frac{n}{L} \rceil, BL, s)$. For each $i \in L$ let

$$S_i = \{x \in n : \tfrac{i}{L} \leq f(x) \leq \tfrac{i+1}{L}\}.$$

There exists an $i \in L$ such that $|S_i| \geq \lceil \frac{n}{L} \rceil$; let S be a subset of this S_i of size exactly $|S| = \lceil \frac{n}{L} \rceil$. Let $\varphi : |S| \to S$ be the enumeration of S; that is, φ is the monotonic increasing bijection from $|S|$ to S. Define $g : |S| \to [0, 1]$ by setting

$$g(x) = L \cdot \left(f(\varphi(x)) - \tfrac{i}{L}\right). \tag{10}$$

By the definition of $c_\alpha(|S|, BL, s)$, there exists a set $T \subset |S|$ such that

$$\mu^s(T) \geq c_\alpha(|S|, BL, s) \quad \text{and} \quad g|_T \in BL\text{-Hölder}^\alpha. \tag{11}$$

Since φ is a non-contractive mapping, from Lemma 2.6, (11) and (10) we obtain that

$$\mu^s(\varphi(T)) \geq \mu^s(T) \geq c_\alpha(|S|, BL, s) \quad \text{and} \quad f|_{\varphi(T)} \in B\text{-Hölder}^\alpha.$$

Thus A can be chosen as $\varphi(T)$, which proves this Lemma. ☐

Proof of Theorem 3.4. We consider α and $s < 1 - \alpha$ to be fixed. Let K be a sufficiently large positive integer. At first we shall prove the theorem for some $B \geq 4K$.

Let us apply Lemma 3.7, we obtain an integer N such that for all $n \geq N$ we have

$$c_\alpha(n, B/n^\alpha, s) \geq \min\left(\left(\tfrac{n}{4K}\right)^s, K \cdot c_\alpha(\lceil \tfrac{n}{2K} \rceil, B/n^\alpha, s)\right) \tag{12}$$

since $B/n^\alpha \geq \frac{4K}{n^\alpha}$ holds as $B \geq 4K$.

Let us choose an integer L such that

$$(2K)^{\frac{\alpha}{1-\alpha}} \leq L \leq \tfrac{1}{2} K^{\frac{1-s}{s}} \tag{13}$$

holds (this can be done if K is sufficiently large since $s < 1 - \alpha$). It is easy to check that the lower bound implies

$$\frac{B}{\lceil \frac{n}{2KL} \rceil^\alpha} \leq \frac{BL}{n^\alpha}. \tag{14}$$

Now apply Lemma 3.8 for the right hand side of (12), we obtain

$$c_\alpha(n, B/n^\alpha, s) \geq \min\left(\left(\tfrac{n}{4K}\right)^s, K \cdot c_\alpha(\lceil \lceil \tfrac{n}{2K} \rceil \tfrac{1}{L} \rceil, BL/n^\alpha, s)\right) \qquad (n \geq N).$$

Since c_α is monotonic increasing in its first and second coordinates, applying (14) we obtain

$$c_\alpha(n, B/n^\alpha, s) \geq \min\left(\left(\tfrac{n}{4K}\right)^s, \ K \cdot c_\alpha(\lceil \tfrac{n}{2KL} \rceil, \frac{B}{\lceil \frac{n}{2KL} \rceil^\alpha}, s)\right) \qquad (n \geq N). \tag{15}$$

Now let n_0 be an arbitrary positive integer and define the sequence (n_i) by

$$n_{i+1} = \lceil \tfrac{n_i}{2KL} \rceil \qquad (i \in \mathbb{N}). \tag{16}$$

Let j be the smallest nonnegative integer for which either

$$c_\alpha(n_j, B/n_j^\alpha, s) \geq \left(\tfrac{n_j}{4K}\right)^s$$

or $n_j \leq N$ holds. Then from (15) we deduce that for all $0 \leq i < j$ we have

$$c_\alpha(n_i, B/n_i^\alpha, s) \geq K \cdot c_\alpha(n_{i+1}, B/n_{i+1}^\alpha, s), \tag{17}$$

and

$$c_\alpha(n_j, B/n_j^\alpha, s) \geq \left(\tfrac{n_j}{4KN}\right)^s, \tag{18}$$

since if $n_j \leq N$, then the right hand side is smaller than 1, which is a trivial lower bound. From (17), (18) and (16) we get

$$c_\alpha(n_0, B/n_0^\alpha, s) \geq K^j \left(\tfrac{n_j}{4KN}\right)^s \geq K^j \left(\tfrac{1}{4KN}\right)^s n_0^s \left(\tfrac{1}{2KL}\right)^{js} = \left(\tfrac{1}{4KN}\right)^s n_0^s \left(\tfrac{K}{2^s K^s L^s}\right)^j \geq \left(\tfrac{1}{4KN}\right)^s n_0^s,$$

provided that $\frac{K}{2^s K^s L^s} \geq 1$, which is equivalent to $L \leq \frac{1}{2} K^{\frac{1-s}{s}}$, which is the upper bound in (13). Thus we have

$$\inf_{n \geq 1} \frac{c_\alpha(n, B/n^\alpha, s)}{n^s} \geq \left(\tfrac{1}{4KN}\right)^s > 0 \tag{19}$$

for all $B \geq 4K$.

Now let $B > 0$ be arbitrary and L be a positive integer so large that $BL \geq 4KL^\alpha$ holds. Using Lemma 3.8 and the fact that c_α is monotonic increasing in its second coordinate, and then (19),

$$\inf_{n \geq 1} \frac{c_\alpha(n, B/n^\alpha, s)}{n^s} \geq \inf_{n \geq 1} \frac{c_\alpha(\lceil \frac{n}{L} \rceil, BL/n^\alpha, s)}{n^s}$$
$$\geq \inf_{n \geq 1} \frac{c_\alpha(\lceil \frac{n}{L} \rceil, 4K/\lceil \frac{n}{L} \rceil^\alpha, s)}{n^s} \geq \inf_{n' \geq 1} \frac{c_\alpha(n', 4K/n'^\alpha, s)}{(n'L)^s} > 0.$$

□

4 The continuous case

4.1 Limit theorems

In the informal overview at the beginning of Section 3 a precise theorem about uniform convergence was stated. We will not prove that theorem for two reasons. On the one hand, it is not sufficient for us, because we also have to deal with functions $f : [0,1] \to \mathbb{R}$ which are not continuous, just measurable. On the other hand, we do not need a theorem in this generality, it is more convenient to prove a similar theorem only for some specific series f_n.

Let $K \subset \mathbb{R}$ be compact, and let $f : K \to \mathbb{R}$ be continuous. Let

$$K_n = \bigcup \{ [\tfrac{i}{n}, \tfrac{i+1}{n}] : K \cap [\tfrac{i}{n}, \tfrac{i+1}{n}] \neq \emptyset, \ \ i \in \mathbb{Z} \},$$

and define $f_n : K_n \to \mathbb{R}$ by setting

$$f_n(x) = f\big(\min(K \cap [\tfrac{i}{n}, \tfrac{i+1}{n}])\big)$$

where i is the largest integer for which $x \in [\frac{i}{n}, \frac{i+1}{n}] \subset K_n$ holds. Thus f_n is piecewise constant on the intervals $[\frac{i}{n}, \frac{i+1}{n}]$.

For $X \subset \mathbb{R}$, let $B(X, r)$ denote the r-neighborhood of the set X.

Lemma 4.1. *Let $0 < s \leq 1$. Suppose that $C_n \subset K_n$ are compact sets with $\mathcal{H}^s_\infty(C_n) \geq \varepsilon$ ($n \in \mathbb{N}$) for some $\varepsilon > 0$. Let C be an accumulation point of (C_n) in the Hausdorff metric. Then $C \subset K$ and $\mathcal{H}^s_\infty(C) \geq \varepsilon$.*

Proof. Since $\limsup K_n = K$, $C \subset K$ is trivial. Suppose indirectly that $\mathcal{H}^s_\infty(C) < \varepsilon$. Then there exists an $r > 0$ such that $\mathcal{H}^s_\infty(B(C, r)) < \varepsilon$ also holds. There exists an n such that $C_n \subset B(C, r)$ (since C is an accumulation point), which contradicts the fact that $\mathcal{H}^s_\infty(C_n) \geq \varepsilon$. □

Lemma 4.2. *Suppose that $C_n \subset K_n$ are compact sets such that $\mathrm{Var}\, f_n|_{C_n} \le B$ for some $B \ge 0$. Let C be an accumulation point of (C_n) in the Hausdorff metric. Then $\mathrm{Var}\, f|_C \le B$ also holds.*

Proof. Let n_j be a sequence of integers such that $C_{n_j} \to C$ in the Hausdorff metric. Let $x_1 < x_2 < \ldots < x_k$ be points in C. Let $\varepsilon > 0$ be arbitrary. There exist an $n = n_j$ and $\delta > 0$ such that $C \subset B(C_n, \delta)$ and $|f(x) - f(y)| < \varepsilon$ if $|x - y| < \delta + \frac{1}{n}$ $(x, y \in K)$.

Let $y_i \in C_n$ be such that $|x_i - y_i| < \delta$ $(i = 1, \ldots, k)$ and $y_1 \le y_2 \le \cdots \le y_k$. By the definition of f_n, there exist $z_i \in K$ such that $f_n(y_i) = f(z_i)$ and $|z_i - y_i| \le \frac{1}{n}$ $(i = 1, \ldots, k)$.

Since $|z_i - x_i| < \delta + \frac{1}{n}$, we have $|f(z_i) - f(x_i)| < \varepsilon$. Using that $\mathrm{Var}\, f_n|_{C_n} \le B$, we have

$$B \ge \sum_{i=1}^{k-1} |f_n(y_i) - f_n(y_{i+1})| = \sum_{i=1}^{k-1} |f(z_i) - f(z_{i+1})|,$$

and thus

$$\sum_{i=1}^{k-1} |f(x_i) - f(x_{i+1})| \le B + 2k\varepsilon.$$

This holds for all $\varepsilon > 0$, thus the total variation of $f|_C$ is also at most B. □

Lemma 4.3. *Let $0 < \alpha \le 1$ and $B > 0$. Suppose that $C_n \subset K_n$ are compact sets such that $f_n|_{C_n} \in B$-Hölder$^\alpha$. Let C be an accumulation point of (C_n) in the Hausdorff metric. Then $f|_C \in B$-Hölder$^\alpha$.*

Proof. Let n_j be a sequence of integers such that $C_{n_j} \to C$ in the Hausdorff metric. Let $x_1, x_2 \in C$, $x_1 \ne x_2$. Let $\varepsilon > 0$ be arbitrary. There exist an $n = n_j$ and $0 < \delta < \varepsilon$ such that $C \subset B(C_n, \delta)$ and $|f(x) - f(y)| < \varepsilon$ if $|x - y| < \delta + \frac{1}{n}$ $(x, y \in K)$.

Let $y_i \in C_n$ be such that $|x_i - y_i| < \delta$ $(i = 1, 2)$. By the definition of f_n, there exists $z_i \in K$ such that $f_n(y_i) = f(z_i)$ and $|z_i - y_i| \le \frac{1}{n}$ $(i = 1, 2)$.

Since $|z_i - x_i| < \delta + \frac{1}{n}$, we have $|f(z_i) - f(x_i)| < \varepsilon$. Because $f_n|_{C_n} \in B$-Hölder$^\alpha$, we have

$$|f(z_1) - f(z_2)| = |f_n(y_1) - f_n(y_2)| \le B|y_1 - y_2|^\alpha,$$

thus

$$|f(x_1) - f(x_2)| \le 2\varepsilon + B|y_1 - y_2|^\alpha$$

and

$$|f(x_1) - f(x_2)| \le 2\varepsilon + B|x_1 - x_2|^\alpha + B(2\varepsilon)^\alpha.$$

Since this holds for all $\varepsilon > 0$, we obtain that $f|_C \in B$-Hölder$^\alpha$. □

We remark that the sets C_n in the previous Lemmas are all contained in a compact interval, hence the sequence (C_n) has an accumulation point in the Hausdorff metric.

4.2 Proof of the main theorems

Let us denote the Lebesgue measure by λ.

Proposition 4.4. *Let $K \subset [0, 1]$ be a compact set of positive Lebesgue measure, $f : K \to \mathbb{R}$ be continuous, and let $s < 1/2$, $B > 0$. There exists a compact set $C \subset [0, 1]$ of Hausdorff dimension at least s such that $\mathrm{Var}\, f|_C \le B$.*

Proposition 4.5. *Let $K \subset [0, 1]$ be a compact set of positive Lebesgue measure, $f : K \to \mathbb{R}$ be continuous, and let $0 < \alpha < 1$, $s < 1 - \alpha$ and $B > 0$. There exists a compact set $C \subset [0, 1]$ of Hausdorff dimension at least s such that $f|_C \in B$-Hölder$^\alpha$.*

Proof of Proposition 4.4. We may suppose without loss of generality that $f(K) \subset [0, 1]$. Let K_n and f_n be defined as above.

Let $\lambda_n = n \cdot \lambda(K_n)$. Then

$$K_n = [\tfrac{\varphi_n(0)}{n}, \tfrac{\varphi_n(0)+1}{n}] \cup \ldots \cup [\tfrac{\varphi_n(\lambda_n-1)}{n}, \tfrac{\varphi_n(\lambda_n-1)+1}{n}]$$

for some integers $\varphi_n(0) < \varphi_n(1) < \ldots < \varphi_n(\lambda_n - 1)$. Note that $\varphi_n : \lambda_n \to \mathbb{N}$ is a non-contractive mapping. Define the function $g_n : \lambda_n \to [0, 1]$ by setting

$$g_n(k) = f_n(\tfrac{\varphi_n(k)}{n}) \qquad (k \in \lambda_n). \tag{20}$$

Let us apply Theorem 3.3 for the functions g_n (for every positive integer n). We obtain some $\varepsilon > 0$ and subsets $A_n \subset \lambda_n$ such that $\mu^s(A_n) \ge \lambda_n^s \varepsilon \ge \lambda(K)^s n^s \varepsilon$ and $\mathrm{Var}\, g_n|_{A_n} \le B$.

Let $C_n = n^{-1} \cdot (\varphi_n(A_n))^*$ (see then definition of $*$ in Lemma 2.4), thus $C_n \subset K_n$. It is easy to see that we have $\operatorname{Var} f_n|_{C_n} = \operatorname{Var} g_n|_{A_n} \leq B$. From Lemma 2.4 we get

$$\mathcal{H}^s_\infty(C_n) = n^{-s}\mathcal{H}^s_\infty((\varphi_n(A_n))^*) \geq n^{-s}2^{-s}\mu^s(\varphi_n(A_n)),$$

and since φ_n is a non-contractive mapping we get from Lemma 2.6 that

$$\mathcal{H}^s_\infty(C_n) \geq n^{-s}2^{-s}\mu^s(A_n) \geq n^{-s}2^{-s}\lambda(K)^s n^s \varepsilon = \lambda(K)^s 2^{-s}\varepsilon.$$

Now choose an accumulation point C of (C_n). We immediately see from Lemma 4.1 that $C \subset K$, $\mathcal{H}^s_\infty(C) \geq \lambda(K)^s 2^{-s}\varepsilon > 0$, thus the Hausdorff dimension of C is at least s; and from Lemma 4.2 we get that $\operatorname{Var} f|_C \leq B$. □

Proof of Proposition 4.5. We only sketch this proof since it is analogous to the proof of Proposition 4.4. The only minor difficulty here is in deducing the Hölder constant "B" of $f_n|_{C_n}$ from the Hölder constant of $g_n|_{A_n}$.

Define the functions $\varphi_n : \lambda_n \to \mathbb{N}$ and $g_n : \lambda_n \to [0,1]$ the same way as above. Now apply Theorem 3.4 to obtain some $\varepsilon > 0$ and subsets $A_n \subset \lambda_n$ with the properties that $\mu^s(A_n) \geq \lambda_n^s \varepsilon \geq \lambda(K)^s n^s \varepsilon$ and $g_n|_{A_n} \in B/n^\alpha$-Hölder$^\alpha$.

Let $C_n = n^{-1} \cdot (\varphi_n(A_n))^* \subset K_n$. From Lemma 2.4 we immediately see that

$$\mathcal{H}^s_\infty(C_n) \geq n^{-s}2^{-s}\mu^s(A_n) \geq n^{-s}2^{-s}\lambda(K)^s n^s \varepsilon = \lambda(K)^s 2^{-s}\varepsilon.$$

On the other hand, we have

Lemma 4.6. *$f_n|_{C_n} \in B2^\alpha$-Hölder$^\alpha$.*

Proof. Let $x, y \in C_n$ be arbitrary. Then $nx, ny \in (\varphi_n(A_n))^*$; that is, there exist $i, j \in A_n$ such that

$$nx \in [\varphi_n(i), \varphi_n(i) + 1/2] \tag{21}$$

and that

$$ny \in [\varphi_n(j), \varphi_n(j) + 1/2]. \tag{22}$$

We have chosen A_n so that the inequality $|g_n(i) - g_n(j)| \leq B/n^\alpha \cdot |i-j|^\alpha$ holds. Using the definition of g_n (see (20)) and that φ_n is non-contractive,

$$|f_n(\tfrac{\varphi_n(i)}{n}) - f_n(\tfrac{\varphi_n(j)}{n})| = |g_n(i) - g_n(j)| \leq B/n^\alpha \cdot |i-j|^\alpha \leq B/n^\alpha \cdot |\varphi_n(i) - \varphi_n(j)|^\alpha.$$

From the definition of f_n (see the beginning of the section), the left hand side is just $|f_n(x) - f_n(y)|$. Thus, using (21) and (22), we obtain

$$|f_n(x) - f_n(y)| \leq B/n^\alpha \cdot |\varphi_n(i) - \varphi_n(j)|^\alpha \leq B/n^\alpha \cdot |2(nx - ny)|^\alpha = B2^\alpha |x-y|^\alpha.$$

□

Now choose an accumulation point C of (C_n). From Lemma 4.1 we deduce that $C \subset K$, $\mathcal{H}^s_\infty(C) \geq \lambda(K)^s 2^{-s}\varepsilon > 0$ (thus C has Hausdorff dimension at least s), and from Lemma 4.3 we get that $f|_C \in B2^\alpha$-Hölder$^\alpha$. This finishes the proof (though we obtained a Hölder constant $B2^\alpha$ instead of B). □

Now we are ready to prove the main theorems.

Proof of Theorem 1.4. There exists a compact set $K \subset [0,1]$ of positive Lebesgue measure such that $f|_K$ is continuous. We may suppose that every non-empty intersection of K with an open interval has positive Lebesgue measure, since we may remove those non-empty intersections from K which are of Lebesgue measure zero (and we need to remove only countably many). Therefore we will be able to use Proposition 4.4 not only in K, but in any non-empty portion of K.

Let $x \in K$, and let $x_n \searrow x$ be a strictly decreasing sequence in K converging fast enough to ensure

$$\sum_{n=1}^{\infty} \sup_{y \in [x, x_n]} |f(y) - f(x)| \leq 1.$$

For each positive integer n, let us apply Proposition 4.4 for the function f restricted to $K \cap [x_{2n+2}, x_{2n}]$ to obtain a compact set $C_n \subset K \cap [x_{2n+2}, x_{2n}]$ of dimension at least $1/2 - 1/n$ such that $\operatorname{Var} f|_{C_n} \leq 2^{-n}$. Let C be the closure of $\cup_n C_n$ (which is $\cup_n C_n \cup \{x\}$). Thus C is of dimension at least $1/2$ and $\operatorname{Var} f|_C \leq 1 + \sum_n 2^{-n} = 2$.

We may choose a compact subset of C of dimension exactly $1/2$ (see e.g. [7]), which finishes the proof. □

Proof of Theorem 1.5. Suppose that K has the same property as in the previous proof; that is, every non-empty intersection of K with an open interval has positive Lebesgue measure. Let us apply Proposition 4.5 for some $0 < s < 1 - \alpha$ to obtain a compact set C' of dimension at least s such that $f|_{C'} \in 1\text{-Hölder}^{\alpha}$. Choose a strictly decreasing sequence (x_n) in C'. Thus f is also Hölder-α with constant 1 on the series (x_n). For each positive integer n, let $\varepsilon_n > 0$ be very small. Now for each positive integer n apply Proposition 4.5 for f restricted to $K \cap [x_n - \varepsilon_n, x_n + \varepsilon_n]$ to obtain a compact set $C_n \subset K \cap [x_n - \varepsilon_n, x_n + \varepsilon_n]$ of dimension at least $1 - \alpha - 1/n$ such that $f|_{C_n} \in 1\text{-Hölder}^{\alpha}$. Let C be the closure of $\cup_n C_n$. Thus C is of dimension at least $1 - \alpha$. It is clear that if the numbers ε_n are chosen to be small enough, then $C = \cup_n C_n \cup \{\lim x_n\}$, and from the continuity of f, that $f|_C \in 2\text{-Hölder}^{\alpha}$.

Again, we may choose a compact subset of C of dimension exactly $1 - \alpha$, which finishes the proof. □

5 Generalizations and open questions

Definition 5.1. The β-variation of a function $f : A \to \mathbb{R}$ (or $f : A \to \mathbb{R}^m$) is defined as

$$\sup\Big\{ \sum_{i=1}^{n-1} |f(x_{i+1}) - f(x_i)|^{\beta} \; : \; x_1 < x_2 < \ldots < x_n, \; x_i \in A \Big\}.$$

In an exactly similar manner as we proved Theorem 3.3 and Theorem 1.4 one can generalize these theorems for bounded β-variations instead of bounded 1-variation.

Theorem 5.2. *Let $f : [0,1] \to \mathbb{R}$ be Lebesgue measurable, $\beta > 0$. There exists a compact set C of Hausdorff dimension $\frac{\beta}{1+\beta}$ such that f has finite β-variation on C.*

This result is sharp, since the methods of M. Elekes in [4] can also be generalized to show that a typical continuous function has infinite β-variation on any set of dimension larger than $\frac{\beta}{1+\beta}$.

Using standard techniques it is straightforward to generalize Theorem 5.2 and Theorem 1.5 to higher dimensional Euclidean spaces. (Namely, one can exploit the fact that it is possible to map a 'large portion' of $\mathbb{R}$ to a 'large portion' of $\mathbb{R}^n$ by a Hölder-$1/n$ mapping so that its inverse is Hölder-n.)

Theorem 5.3. *Let $f : \mathbb{R} \to \mathbb{R}^m$ be Lebesgue measurable, $\beta > 0$. There exists a compact set C of Hausdorff dimension $\frac{\beta}{m+\beta}$ such that f has finite β-variation on C.*

Theorem 5.4. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lebesgue measurable and let $0 < \alpha < \frac{n}{m}$. There exists a compact set C of Hausdorff dimension $n - m\alpha$ such that f is Hölder-α on C.*

These theorems (the stated dimensions) are again sharp, for all β, m and n.

We have shown that every $\mathbb{R} \to \mathbb{R}$ Borel function is of bounded variation on some compact set of Hausdorff dimension 1/2. However, we do not know anything about the possible (1/2-dimensional) Hausdorff measure of such sets.

Question 5.5. Can we find for every Borel function $f : [0,1] \to \mathbb{R}$ a Borel set B of positive 1/2-dimensional Hausdorff measure such that f restricted to B is of bounded variation?

Question 5.6. Does there exist a Borel function $f : [0,1] \to \mathbb{R}$ such that if f is of bounded variation on some Borel set B, then B has zero/finite/σ-finite 1/2-dimensional Hausdorff measure?

The analogous questions for the Hölder-α property are also open.

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