

The isomorphism problem of
Hausdorff measures
and Hölder restrictions of functions

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Chapter 1

Introduction

1.1 The questions

It was an unsolved problem for several years whether Hausdorff measures of different dimensions can be Borel isomorphic. This problem was attributed to B. Weiss and was popularized by D. Preiss. Let \mathcal{B} denote the Borel σ -algebra of \mathbb{R} , and let \mathcal{H}^d denote the d -dimensional Hausdorff measure; then the exact question reads as follows.

Question 1. Let $0 \leq s, t \leq 1$ and $s \neq t$.

- (i) Can the measure spaces $(\mathbb{R}, \mathcal{B}, \mathcal{H}^s)$ and $(\mathbb{R}, \mathcal{B}, \mathcal{H}^t)$ be isomorphic?
- (ii) Does there exist a Borel bijection $f : \mathbb{R} \rightarrow \mathbb{R}$ such that for every Borel set B ,

$$0 < \mathcal{H}^s(B) < \infty \iff 0 < \mathcal{H}^t(f(B)) < \infty?$$

The two parts are not equivalent but it is easy to see that a negative answer to (ii) implies a negative answer to (i).

It is important to make the distinction that we are looking for *Borel* isomorphisms only. M. Elekes [4] proved that if we assume the continuum hypothesis then the measure spaces $(\mathbb{R}, \mathcal{M}_{\mathcal{H}^s}, \mathcal{H}^s)$ and $(\mathbb{R}, \mathcal{M}_{\mathcal{H}^t}, \mathcal{H}^t)$ are isomorphic whenever $s, t \in (0, 1)$, where $\mathcal{M}_{\mathcal{H}^d}$ denotes the σ -algebra of the sets which are measurable with respect to \mathcal{H}^d .

Remark 1.1.1. In fact, the bijection $f : \mathbb{R} \rightarrow \mathbb{R}$ that M. Elekes constructed (assuming the continuum hypothesis) satisfies that

- if $B \subset \mathbb{R}$ is Borel and $\mathcal{H}^s(B) < \infty$, then $f(B)$ is Borel and $\mathcal{H}^s(B) = \mathcal{H}^t(f(B))$;
- if $f(B)$ is Borel and $\mathcal{H}^t(f(B)) < \infty$, then B is Borel and $\mathcal{H}^s(B) = \mathcal{H}^t(f(B))$.

However, this map f is not Borel measurable.

M. Elekes, aiming to solve the original problem (Question 1), raised the following question [4].

Question 2. Let $0 < \alpha < 1$. Can we find for every Borel (or continuous, or typical continuous) function $f : [0, 1] \rightarrow \mathbb{R}$ a Borel set $B \subset [0, 1]$ of positive Hausdorff dimension such that f restricted to B is Hölder continuous of exponent α ?

How is this question related to the previous one? Suppose that we have an answer to Question 2 saying that (for some fixed α) for every Borel function f there exists a Borel set B of dimension β such that f is Hölder- α on B . As it is well-known, this implies that $f(B)$ has dimension at most β/α . It is easy to see that this would answer (both parts of) Question 1 in the negative for those s and t for which $0 \leq s < \beta < \beta/\alpha < t \leq 1$ holds.

According to a theorem of P. Humke and M. Laczkovich [6], a typical continuous function $f : [0, 1] \rightarrow \mathbb{R}$ is not monotonic on any set of positive Hausdorff dimension. Since every function of bounded variation is the sum of two monotonic functions, this theorem motivated M. Elekes to raise an analogue of Question 2.

Question 3. Can we find for every Borel (or continuous, or typical continuous) function $f : [0, 1] \rightarrow \mathbb{R}$ a Borel set $B \subset [0, 1]$ of positive Hausdorff dimension such that f restricted to B is of bounded variation? Can we even find such a set of dimension $1/2$?

This problem has also been circulated by D. Preiss, and a similar question was already asked by P. Humke and M. Laczkovich, see also Z. Buczolich [2, 3].

M. Elekes gave partial answers to Question 3 and Question 2 in [4]. He proved that a typical continuous function $f : [0, 1] \rightarrow \mathbb{R}$ is not of bounded variation on any set of Hausdorff dimension larger than $1/2$. Regarding Question 2, he also gave an upper bound for the possible dimension by showing that for every $0 < \alpha \leq 1$, a typical continuous function is not Hölder- α on any set of dimension larger than $1 - \alpha$.

1.2 Restrictions of functions

We answer Question 2 and Question 3 by proving the following theorems. (In this chapter all theorems are numbered as they will appear in the following chapters.)

Theorem 2.1.1. *Let $f : [0, 1] \rightarrow \mathbb{R}$ be Lebesgue measurable. Then there exists a compact set $C \subset [0, 1]$ of Hausdorff dimension $1/2$ such that $f|_C$ is of bounded variation.*

Theorem 2.1.2. *Let $f : [0, 1] \rightarrow \mathbb{R}$ be Lebesgue measurable and let $0 < \alpha < 1$. Then there exists a compact set $C \subset [0, 1]$ of Hausdorff dimension $1 - \alpha$ such that $f|_C$ is a Hölder- α function.*

That is, the dimension bounds found by M. Elekes are sharp.

These theorems will be proved in Chapter 2. To prove Theorem 2.1.1, first we will define discrete Hausdorff pre-measures on \mathbb{Z} (the integers). Using this notion we will be able to formalize and solve a discrete (quantitative) version of the problem. Then suitable limit theorems will yield Theorem 2.1.1. It is possible to prove Theorem 2.1.2 in exactly the same way. However, we will present a simple proof instead, using a theorem of P. Mattila about Hausdorff dimensions of plane sections.

We will also mention some generalizations of Theorems 2.1.1 and 2.1.2.

Remark 1.2.1. Theorems 2.1.1 and 2.1.2 belong to the family of *restriction theorems*. The setting of a restriction theorem usually is the following. Given some function f from some class X , one tries to find a large set A such that $f|_A$ belongs to some other (nice) class Y . Here largeness usually means that A is infinite, uncountable, perfect, not porous, or A is of positive measure or of second category. It is interesting that for the above questions of M. Elekes, the proper notion of largeness is Hausdorff dimension. We refer to the survey article of J. B. Brown [1] on restriction theorems and to the references therein.

Remark 1.2.2. Notice that if $A \subset \mathbb{R}$ and $f : A \rightarrow \mathbb{R}$ is a given function, then there exists a function $g : \mathbb{R} \rightarrow \mathbb{R}$ extending f (that is, $g|_A = f$) such that the total variation of g and f are equal and that f is Hölder- α if and only if g is Hölder- α ($0 < \alpha \leq 1$). (Given f , one can easily define g on the closure of A , and then the linear extension works.) This yields that for every function $f : [0, 1] \rightarrow \mathbb{R}$ and $\beta \in [0, 1]$ the following are equivalent:

- (i) there exists a set A of dimension at least β such that $f|_A$ is of bounded variation;
- (ii) there exists a function $g : [0, 1] \rightarrow \mathbb{R}$ of bounded variation such that the set $[f = g]$ (that is, $\{x : f(x) = g(x)\}$) has dimension at least β .

The same equivalence holds for the Hölder- α property. Thus Questions 3 and 2 and Theorems 2.1.1 and 2.1.2 could have been formulated equivalently corresponding to (ii) as well.

1.3 Borel maps and Hausdorff dimension

As explained above, Theorem 2.1.2 gives some partial results on the isomorphism problem of Hausdorff measures (Question 1). For example, applying the theorem for $\alpha = 2/3$ implies that the Hausdorff measures of dimension $1/3 - \varepsilon$ and $1/2 + \varepsilon$ cannot be Borel isomorphic. However, this approach does not seem to answer Question 1 in its full generality.

In Chapter 3 we will show the following.

Theorem 3.1.2. *Let $0 \leq d \leq 1$ and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be Borel measurable. Then there exists a compact set C of Hausdorff dimension d such that $f(C)$ is of Hausdorff dimension at most d .*

Clearly, this theorem immediately implies that the Hausdorff measures of different dimensions cannot be Borel isomorphic; that is, we answer (both parts of) Question 1 in the negative. The proof of Theorem 3.1.2 is based on two types of random constructions. One of them can be used to obtain a random Cantor set of dimension at most d almost surely; and the other to obtain a random compact set of dimension at least d almost surely.

We will also prove the following generalization of Theorem 3.1.2.

Theorem 3.1.5. *Let $D \subset \mathbb{R}^n$ be a Borel set and let $f : D \rightarrow \mathbb{R}^m$ be Borel measurable. Then for every $0 \leq d \leq 1$ there exists a Borel set $A \subset D$ such that $\dim A = d \cdot \dim D$ and $\dim f(A) \leq d \cdot \dim f(D)$.*

1.4 Borel maps and Hausdorff measures

After we have shown that Hausdorff measures of different dimensions are not Borel isomorphic, another question arises. Can it happen that \mathcal{H}^s and $2 \cdot \mathcal{H}^s$ are Borel isomorphic? That is, does there exist a Borel bijection f such that $2 \cdot \mathcal{H}^s(B) = \mathcal{H}^s(f(B))$ for every Borel set B ? Of course, if we consider \mathbb{R} (and $\mathbb{R} \rightarrow \mathbb{R}$ Borel bijections), then the answer is positive: similarities of ratio $2^{1/s}$ realize the isomorphism. However, if we consider the unit interval $[0, 1]$ only, then the question is highly non-trivial. We will answer this question in Chapter 4 by showing the following.

Theorem 4.5.6. *Let $f : [0, 1] \rightarrow [0, 1]$ be Borel measurable. Then for every $0 < s < 1$ there exists a compact set $C \subset [0, 1]$ such that $\mathcal{H}^s(C) = 1$ and $\mathcal{H}^s(f(C)) \leq 1$.*

The proof of this theorem is similar to the proof of Theorem 3.1.2. That is, we will use two types of random constructions. However, to prove that the random

set produced by one of the constructions is sufficiently large, requires much more effort than in the analogous proof in Chapter 3. This is the reason why we discuss Theorem 3.1.2 separately, despite the fact that Theorem 4.5.6 is a stronger statement.

In Chapter 4 we will also prove several related theorems. Among others, we will show that if $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a Borel bijection which preserves the s -dimensional Hausdorff measure (for some $0 < s < n$), then it also preserves the Lebesgue measure (Theorem 4.5.21).

1.5 Notation and definitions

We say that the real function f is of bounded variation on the set A if f restricted to A is a function of bounded variation. We say that the real function f is Hölder continuous of exponent α (or briefly Hölder- α) on the set A if $f|_A$ is Hölder- α ; that is, there exists a real number $B > 0$ such that for every $x, y \in A$, $|f(x) - f(y)| \leq B|x - y|^\alpha$.

Given $\emptyset \neq A \subset \mathbb{R}$ and $f : A \rightarrow \mathbb{R}$, we denote the total variation of f by $\text{Var } f$; that is,

$$\text{Var } f = \sup \left\{ \sum_{i=1}^{n-1} |f(x_{i+1}) - f(x_i)| : n \geq 1, x_1 < x_2 < \dots < x_n, x_i \in A \right\}.$$

Given $\emptyset \neq A \subset \mathbb{R}$ and $f : A \rightarrow \mathbb{R}$, we say that $f \in B$ -Hölder $^\alpha$ if

$$\forall x, y \in A \quad |f(x) - f(y)| \leq B|x - y|^\alpha.$$

The word “typical” is used in the Baire category sense.

For $x \in \mathbb{R}$, we denote by $[x]$ the ceiling of x , that is, the smallest integer not smaller than x .

We denote by \mathbb{N} the set of non-negative integers. As usual in set theory, we identify each $n \in \mathbb{N}$ with the set of its predecessors: $n = \{0, 1, \dots, n-1\}$.

The Lebesgue measure on \mathbb{R} and on any Euclidean space \mathbb{R}^n will be denoted by λ . It will be always clear from the context which Lebesgue measure we are using.

We denote the diameter of a set $U \subset \mathbb{R}^n$ by $\text{diam } U$.

Let $\mathcal{H}_\infty^s(A)$ denote the s -dimensional Hausdorff pre-measure of the set $A \subset \mathbb{R}$; that is,

$$\mathcal{H}_\infty^s(A) = \inf \left\{ \sum_{i=1}^{\infty} (\text{diam } I_i)^s : A \subset \bigcup_{i=1}^{\infty} I_i \right\}.$$

Let $\mathcal{H}^s(A)$ denote the s -dimensional Hausdorff measure of $A \subset \mathbb{R}^n$; that is,

$$\mathcal{H}^s(A) = \lim_{\delta \searrow 0} \mathcal{H}_\delta^s(A),$$

where

$$\mathcal{H}_\delta^s(A) = \inf \left\{ \sum_{i=1}^{\infty} (\text{diam } U_i)^s : A \subset \bigcup_{i=1}^{\infty} U_i, \text{diam } U_i \leq \delta \right\}.$$

The Hausdorff dimension of a set A is then defined as

$$\dim A = \inf \{s > 0 : \mathcal{H}^s(A) = 0\}.$$

By dimension we always mean the Hausdorff dimension.

We call an interval non-trivial if it has positive length. The diameter of an interval I will sometimes be denoted by $|I|$. However, for finite sets S , $|S|$ will denote the number of elements of S . For multi-indices i , $|i|$ will denote the length of i . The intended interpretation will be always clear from the context.

If μ is a Borel measure on \mathbb{R} , we will denote its support by $\text{supp } \mu$; that is,

$$\text{supp } \mu = \{x \in \mathbb{R} : \forall r > 0 \mu((x - r, x + r)) > 0\}.$$

We say that μ is supported on K if $\text{supp } \mu \subset K$.

Probability will be denoted by \mathbb{P} . The expected value of a random variable X will be denoted by $\mathbb{E}X$ or $\mathbb{E}(X)$. Conditional expectation will be denoted by $\mathbb{E}(X|A)$ (where A is an event of positive probability). We denote the indicator variable by $\mathbb{1}$; that is, $\mathbb{1}_A = 1$ if the event A is satisfied and zero otherwise.

Let $f : D \rightarrow \mathbb{R}^m$ be an injective map defined on $D \subset \mathbb{R}^n$. For any set $A \subset \mathbb{R}^n$, we define $f(A)$ to be $f(A \cap D)$. (If we regard f as $(f^{-1})^{-1}$, this is the generally accepted notation.)

Let μ be a Borel measure on \mathbb{R}^n (for example λ or \mathcal{H}^s). Let $D \subset \mathbb{R}^n$ be a Borel set, and let $f : D \rightarrow \mathbb{R}^n$ be a Borel mapping. We say that f preserves the measure μ , if for every Borel set $B \subset f(D)$ we have $\mu(f^{-1}(B)) = \mu(B)$. (Note that this implies that we have $\mu(f^{-1}(B)) \leq \mu(B)$ for every Borel set $B \subset \mathbb{R}^n$.)

We will use many times in our proofs (without explicitly stating) that the image of a Borel set by a Borel mapping is analytic, thus Lebesgue and \mathcal{H}^s -measurable. We will also use the fact that the image of a Borel set by an injective Borel mapping is Borel. We will also need that if $A \subset \mathbb{R}^n$ is analytic and $\mathcal{H}^s(A) > 1$, then there exists a compact set $C \subset A$ such that $\mathcal{H}^s(C) = 1$. See [10, Theorem 8.13] or [5, Theorem 471S].

Chapter 2

Hölder restrictions and restrictions of bounded variation

2.1 Outline

As we stated in the Introduction (§1.2), our goal here is to answer Question 2 and Question 3 by proving the following theorems.

Theorem 2.1.1. *Let $f : [0, 1] \rightarrow \mathbb{R}$ be Lebesgue measurable. Then there exists a compact set $C \subset [0, 1]$ of Hausdorff dimension $1/2$ such that $f|_C$ is of bounded variation.*

Theorem 2.1.2. *Let $f : [0, 1] \rightarrow \mathbb{R}$ be Lebesgue measurable and let $0 < \alpha < 1$. Then there exists a compact set $C \subset [0, 1]$ of Hausdorff dimension $1 - \alpha$ such that $f|_C$ is a Hölder- α function.*

From the results of M. Elekes [4] it follows that these theorems are sharp.

The chapter will be organized as follows. First, in §2.2, we define discrete Hausdorff pre-measures on the integers. Using this notion we formalize and solve a discrete (quantitative) version of Theorem 2.1.1 in §2.3. Then in §2.4 we prove suitable “limit theorems”, and finally deduce Theorem 2.1.1. In §2.5 we present a simple proof of Theorem 2.1.2, using a theorem of P. Mattila about Hausdorff dimensions of plane sections. Finally, in §2.6, we mention a variant of Theorem 2.1.1 regarding generalized variations and we extend our theorems to Euclidean spaces.

We note that Theorem 2.1.2 can be proved in exactly the same way as Theorem 2.1.1. Such a proof of Theorem 2.1.2 (and the proof of Theorem 2.1.1 presented here) can be found in [9].

2.2 Discrete Hausdorff pre-measure

We define the discrete Hausdorff pre-measure on the subsets of the set of integers \mathbb{Z} . The covering sets will be intervals $I \subset \mathbb{Z}$, and here by interval we mean a set of finitely many consecutive integers. By $|I|$ we denote the number of elements of I .

Let $d(X, Y)$ denote the usual distance of X and Y ($X, Y \subset \mathbb{R}$). If X or Y is empty, then we define their distance to be ∞ .

Definition 2.2.1. Let $0 < s \leq 1$. The discrete Hausdorff pre-measure of dimension s is the function $\mu^s : \mathcal{P}(\mathbb{Z}) \rightarrow [0, \infty]$ defined by

$$\mu^s(A) = \min \left\{ \sum_{I \in \mathcal{I}} |I|^s : \mathcal{I} \text{ is a collection of intervals of } \mathbb{Z} \text{ such that } A \subset \bigcup \mathcal{I} \right\}.$$

It is reasonable to call μ^s a pre-measure since it is subadditive.

Lemma 2.2.2. Let $0 < s \leq 1$ and $A, B \subset \mathbb{Z}$. Then

$$\mu^s(A \cup B) \leq \mu^s(A) + \mu^s(B). \quad \square$$

We also have a lower bound of $\mu^s(A \cup B)$.

Lemma 2.2.3. Let $0 < s \leq 1$ and $A, B \subset \mathbb{Z}$. Then

$$\mu^s(A \cup B) \geq \min \left(d(A, B)^s, \mu^s(A) + \mu^s(B) \right).$$

Proof. Consider a covering of $A \cup B$ by intervals of integers. If there is an interval of size at least $d(A, B)$ then the inequality clearly holds. Suppose that every interval is of size at most $d(A, B)$. Then each interval can intersect either A or B but not both, so we can split the covering into two parts to cover A and to cover B , which corresponds to the case $\mu^s(A \cup B) \geq \mu^s(A) + \mu^s(B)$. \square

The following statement connects μ^s to the (real) Hausdorff pre-measure \mathcal{H}_∞^s .

Lemma 2.2.4. Let $0 < s \leq 1$. For a set $A \subset \mathbb{Z}$, let us define

$$A^* \stackrel{\text{def}}{=} \bigcup_i \left\{ \left[i, i + \frac{1}{2} \right] : i \in A \right\}.$$

Then

$$\mathcal{H}_\infty^s(A^*) \leq \mu^s(A) \leq 2^s \cdot \mathcal{H}_\infty^s(A^*).$$

Proof. We may suppose that A is finite (that is, bounded), otherwise $\mathcal{H}_\infty^s(A^*) = \mu^s(A) = \infty$.

We immediately obtain the inequality $\mathcal{H}_\infty^s(A^*) \leq \mu^s(A)$ if we change each covering interval $I \subset \mathbb{Z}$ of A to the interval $[\min I, \max I + 1]$.

Since A^* is compact, it is enough to consider finite coverings of A^* with closed intervals to calculate $\mathcal{H}_\infty^s(A^*)$. Notice that we may suppose that the covering intervals are disjoint, since $(a + b)^s \leq a^s + b^s$ for every $a, b \geq 0$. Hence we may also suppose that all the intervals covering A^* are of the form $[n, n + l + \frac{1}{2}]$ (of length $l + \frac{1}{2}$) for some integer n and $l \in \mathbb{N}$. Hence one can cover A by the corresponding intervals $\{n, n + 1, \dots, n + l\}$ of size $l + 1$. Since $(l + 1)^s \leq 2^s(l + \frac{1}{2})^s$ for every $l \in \mathbb{N}$, we obtain the inequality $\mu^s(A) \leq 2^s \cdot \mathcal{H}_\infty^s(A^*)$. \square

Definition 2.2.5. Let $A \subset \mathbb{Z}$. We say that a mapping $\varphi : A \rightarrow \mathbb{Z}$ is *non-contractive* if $|\varphi(x) - \varphi(y)| \geq |x - y|$ for every $x, y \in A$.

In the sequel we will use the following observation many times.

Lemma 2.2.6. *If $\varphi : A \rightarrow \mathbb{Z}$ is non-contractive, then $\mu^s(\varphi(A)) \geq \mu^s(A)$.* \square

The proof is left to the reader.

2.3 Bounded variation — discrete version

2.3.1 Overview of the proof

Before we start we give an informal overview of the proof of Theorem 2.1.1.

So let $f : [0, 1] \rightarrow \mathbb{R}$ be measurable. Then f is continuous on some compact set of positive measure. Let us just suppose that f is continuous on the whole interval $[0, 1]$, it will not make much difference. We would like to prove that f possesses the property that there exists a set $C \subset [0, 1]$ of large dimension such that $f|_C$ has finite variation. The key observation is that this property (or at least a quantitative version of this property) goes through uniform convergence. That is, if some (not necessarily continuous) functions f_n converge uniformly to f , and there exist compact sets C_n such that

$$\mathcal{H}_\infty^s(C_n) \geq \varepsilon \quad \text{and} \quad \text{Var } f_n|_{C_n} \leq B$$

for some $\varepsilon > 0$ and finite B , then there exists a compact set C such that

$$\mathcal{H}_\infty^s(C) \geq \varepsilon \quad \text{and} \quad \text{Var } f|_C \leq B. \tag{2.1}$$

(Note that $\mathcal{H}_\infty^s(C) > 0$ implies that the dimension of C is at least s .) Hence it is enough to show that this (quantitative) property holds for a dense family of functions. We choose the family of those functions $g : [0, 1] \rightarrow \mathbb{R}$ which are piecewise constant on the intervals $[\frac{i}{n}, \frac{i+1}{n})$ ($i = 0, \dots, n-1$). But for simplicity, we will deal with the discrete functions $h : n \rightarrow \mathbb{R}$ related to g by $h(i) = g(\frac{i}{n})$ ($i = 0, \dots, n-1$).

Now it is not difficult (using Lemma 2.2.4) to relate the following two statements:

- (i) there exists a compact set $C \subset [0, 1]$ such that $\mathcal{H}_\infty^s(C)$ is large and $\text{Var } g|_C$ is small;
- (ii) there exists a set $A \subset n$ such that $\mu^s(A)$ is large and $\text{Var } h|_A$ is small.

Thus we only need to show that statement (ii) (after properly formulated) holds for all n and all functions $h : n \rightarrow \mathbb{R}$, when $s = 1/2$. (Unfortunately, we can show this for any fixed $s < 1/2$ only, but this will be enough to prove Theorem 2.1.1.) In some sense, the proof will go by induction on n . This statement is what we will formulate precisely and prove in this section.

2.3.2 Formalizing the discrete problem

In the following definition n is a positive integer, B is a positive real number, and $s, \alpha \in (0, 1]$.

Definition 2.3.1.

$$b(n, B, s) = \min_{f:n \rightarrow [0,1]} \max_{A \subset n} \{ \mu^s(A) : \text{Var } f|_A \leq B \}.$$

Notice that b is monotone increasing both in n and in B . Clearly, $b(n, B, s) \geq 1$ for all n , since the μ^s -measure of a single point is 1.

The discrete analogue of Theorem 2.1.1 is the following.

Theorem 2.3.2. *For every $0 < s < 1/2$ and $B > 0$,*

$$\inf_{n \geq 1} \frac{b(n, B, s)}{n^s} > 0.$$

Note that the denominator n^s is present because, when we exchange a function $g : [0, 1] \rightarrow \mathbb{R}$ for the function $h : n \rightarrow \mathbb{R}$ (as in the informal overview at the beginning of this section), there is a scaling by a factor of n , and this changes s -dimensional measures by a factor of n^s .

2.3.3 Solution of the discrete problem

First we state and prove two main “induction steps”, and then what remains, will be just calculations.

Lemma 2.3.3. *Fix a positive integer K . Then for every $s < 1/2$ and $B > 0$*

$$b(n, KB + K - 1, s) \geq \min \left(\left(\frac{n}{4K} \right)^s, K \cdot b\left(\lceil \frac{n}{2K} \rceil, B, s\right) \right)$$

for every $n \in \mathbb{N}$ large enough (depending only on K).

Proof. Let us choose K intervals of integers I_1, \dots, I_K inside $n = \{0, 1, \dots, n-1\}$ of size $\lceil \frac{n}{2K} \rceil$ such that the distance of any two of them is at least $\frac{n}{4K}$; clearly this can be done if n is sufficiently large (depending on K).

Fix any function $f : n \rightarrow [0, 1]$. We have to find a set $A \subset n$ of large μ^s -measure such that $\text{Var } f|_A \leq KB + K - 1$. For each interval I_j , consider the function $f|_{I_j}$. Since $|I_j| = \lceil \frac{n}{2K} \rceil$, by the definition of $b(\lceil \frac{n}{2K} \rceil, B, s)$ we can find a set $A_j \subset I_j$ such that

$$\text{Var } f|_{A_j} \leq B \quad \text{and} \quad \mu^s(A_j) \geq b\left(\lceil \frac{n}{2K} \rceil, B, s\right).$$

Put $A = \cup_{j=1}^K A_j$. Applying Lemma 2.2.3 inductively to the sets A_j we get

$$\mu^s(A) \geq \min \left(\left(\frac{n}{4K} \right)^s, K \cdot b\left(\lceil \frac{n}{2K} \rceil, B, s\right) \right). \quad (2.2)$$

Since $\text{Var } f|_A \leq \sum_{j=1}^K \text{Var } f|_{A_j} + (K-1) \leq KB + K - 1$, (2.2) instantly gives Lemma 2.3.3. \square

Lemma 2.3.4. *For each positive integer L ,*

$$b(n, B, s) \geq b\left(\lceil \frac{n}{L} \rceil, BL, s\right).$$

Proof. Fix any function $f : n \rightarrow [0, 1]$. We have to find a set $A \subset n$ such that $\text{Var } f|_A \leq B$ and that $\mu^s(A) \geq b(\lceil \frac{n}{L} \rceil, BL, s)$. For each $i \in L$ let

$$S_i = \left\{ x \in n : \frac{i}{L} \leq f(x) \leq \frac{i+1}{L} \right\}.$$

There exists an $i \in L$ such that $|S_i| \geq \lceil \frac{n}{L} \rceil$; let S be a subset of this S_i of size exactly $|S| = \lceil \frac{n}{L} \rceil$. Let $\varphi : |S| \rightarrow S$ be the enumeration of S ; that is, φ is the monotone increasing bijection from $|S|$ to S . Thus φ is a non-contractive mapping. Define $g : |S| \rightarrow [0, 1]$ by setting

$$g(x) = L \cdot \left(f(\varphi(x)) - \frac{i}{L} \right). \quad (2.3)$$

By the definition of $b(|S|, BL, s)$, there exists a set $T \subset |S|$ such that

$$\mu^s(T) \geq b(|S|, BL, s) \quad \text{and} \quad \text{Var } g|_T \leq BL.$$

Using Lemma 2.2.6 and (2.3),

$$\mu^s(\varphi(T)) \geq \mu^s(T) \geq b(|S|, BL, s) \quad \text{and} \quad \text{Var } f|_{\varphi(T)} = \frac{1}{L} \cdot \text{Var } g|_T \leq B.$$

Thus A can be chosen as $\varphi(T)$, which proves this Lemma. \square

Proof of Theorem 2.3.2. We consider $s < 1/2$ to be fixed. Let K be a sufficiently large positive integer, in fact, let $K > 2^{2s}K^{2s}$ hold. First we will prove the Theorem for $B = 2K - 1$.

Let us apply Lemma 2.3.3 with $B = 1$. We obtain an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$b(n, 2K - 1, s) \geq \min \left(\left(\frac{n}{4K} \right)^s, K \cdot b(\lceil \frac{n}{2K} \rceil, 1, s) \right).$$

Now apply Lemma 2.3.4 to the right hand side with $L = 2K - 1$. We obtain

$$b(n, 2K - 1, s) \geq \min \left(\left(\frac{n}{4K} \right)^s, K \cdot b(\lceil \frac{\lceil \frac{n}{2K} \rceil}{2K - 1} \rceil, 2K - 1, s) \right) \quad (n \geq N).$$

Since b is monotone increasing in its first coordinate,

$$b(n, 2K - 1, s) \geq \min \left(\left(\frac{n}{4K} \right)^s, K \cdot b(\lceil \frac{n}{2K(2K-1)} \rceil, 2K - 1, s) \right) \quad (n \geq N). \quad (2.4)$$

Fix an arbitrary positive integer n_0 , and define the sequence

$$n_{i+1} = \lceil \frac{n_i}{2K(2K-1)} \rceil \quad (i \in \mathbb{N}). \quad (2.5)$$

Let j be the smallest nonnegative integer for which either

$$b(n_j, 2K - 1, s) \geq \left(\frac{n_j}{4K} \right)^s$$

or $n_j \leq N$ holds. Thus from (2.4) we obtain

$$b(n_i, 2K - 1, s) \geq K \cdot b(n_{i+1}, 2K - 1, s) \quad (0 \leq i < j), \quad (2.6)$$

and

$$b(n_j, 2K - 1, s) \geq \left(\frac{n_j}{4KN} \right)^s, \quad (2.7)$$

since if $n_j \leq N$, then the right hand side is smaller than 1, which is a trivial lower

bound. Thus from (2.6) and (2.7) we get

$$b(n_0, 2K - 1, s) \geq K^j \cdot \left(\frac{n_j}{4KN}\right)^s.$$

Using (2.5) we obtain the lower bound

$$\begin{aligned} b(n_0, 2K - 1, s) &\geq K^j \cdot \left(\frac{n_j}{4KN}\right)^s \geq K^j \cdot \left(\frac{1}{4KN}\right)^s n_0^s \left(\frac{1}{2K(2K-1)}\right)^{js} \\ &= \left(\frac{1}{4KN}\right)^s n_0^s \left(\frac{K}{2^s K^s (2K-1)^s}\right)^j \geq \left(\frac{1}{4KN}\right)^s n_0^s \end{aligned} \quad (2.8)$$

provided that $\frac{K}{2^s K^s (2K-1)^s} > 1$, which clearly holds since K was chosen so that $K > 2^{2s} K^{2s}$ holds. Since n_0 was arbitrary, from (2.8) we immediately obtain that

$$\inf_{n \geq 1} \frac{b(n, 2K - 1, s)}{n^s} \geq \left(\frac{1}{4KN}\right)^s > 0. \quad (2.9)$$

Now let $B > 0$ be arbitrary. Let $L \in \mathbb{N}$ be so large that $BL \geq 2K - 1$ holds. Using Lemma 2.3.4, the fact that b is monotone increasing in its second coordinate, and then (2.9),

$$\begin{aligned} \inf_{n \geq 1} \frac{b(n, B, s)}{n^s} &\geq \inf_{n \geq 1} \frac{b(\lceil \frac{n}{L} \rceil, BL, s)}{n^s} \geq \inf_{n \geq 1} \frac{b(\lceil \frac{n}{L} \rceil, 2K - 1, s)}{n^s} \\ &\geq \inf_{n' \geq 1} \frac{b(n', 2K - 1, s)}{(n'L)^s} > 0. \quad \square \end{aligned}$$

2.4 Bounded variation — the continuous case

2.4.1 Limit theorems

The informal overview in §2.3.1 contains a precise statement (2.1) about uniform convergence. We will not prove that statement for two reasons. On the one hand, it is not sufficient for us, because we also have to deal with functions $f : [0, 1] \rightarrow \mathbb{R}$ which are not continuous, just measurable. On the other hand, we do not need the statement in this generality, since it is more convenient to prove a similar theorem for some specific sequence f_n only.

Let $K \subset \mathbb{R}$ be compact, and let $f : K \rightarrow \mathbb{R}$ be continuous. Let

$$K_n = \bigcup \left\{ \left[\frac{i}{n}, \frac{i+1}{n} \right] : K \cap \left[\frac{i}{n}, \frac{i+1}{n} \right] \neq \emptyset, i \in \mathbb{Z} \right\},$$

and define $f_n : K_n \rightarrow \mathbb{R}$ by setting

$$f_n(x) = f\left(\min\left(K \cap \left[\frac{i}{n}, \frac{i+1}{n}\right]\right)\right)$$

where i is the largest integer for which $x \in [\frac{i}{n}, \frac{i+1}{n}] \subset K_n$ holds. Thus f_n is constant on each of the intervals $[\frac{i}{n}, \frac{i+1}{n})$.

For $X \subset \mathbb{R}$, let $B(X, r)$ denote the r -neighborhood of the set X .

Lemma 2.4.1. *Let $0 < s \leq 1$. Suppose that $C_n \subset K_n$ are compact sets with $\mathcal{H}_\infty^s(C_n) \geq \varepsilon$ ($n \in \mathbb{N}$) for some $\varepsilon > 0$. Let C be an accumulation point of (C_n) in the Hausdorff metric. Then $C \subset K$ and $\mathcal{H}_\infty^s(C) \geq \varepsilon$.*

Proof. Since $\limsup K_n \stackrel{\text{def}}{=} \bigcap_N \bigcup_{n=N}^\infty K_n = K$, $C \subset K$ is trivial. Suppose to the contrary that $\mathcal{H}_\infty^s(C) < \varepsilon$. Then there exists an $r > 0$ such that $\mathcal{H}_\infty^s(B(C, r)) < \varepsilon$ also holds. There exists an n such that $C_n \subset B(C, r)$ (since C is an accumulation point), which contradicts the fact that $\mathcal{H}_\infty^s(C_n) \geq \varepsilon$. \square

Lemma 2.4.2. *Suppose that $C_n \subset K_n$ are compact sets such that $\text{Var } f_n|_{C_n} \leq B$ for some $B \geq 0$. Let C be an accumulation point of (C_n) in the Hausdorff metric. Then $\text{Var } f|_C \leq B$ also holds.*

Proof. We know from the previous proof that $C \subset K$. Let n_j be a sequence of integers such that $C_{n_j} \rightarrow C$ in the Hausdorff metric. Let $x_1 < x_2 < \dots < x_k$ be points in C . Let $\varepsilon > 0$ be arbitrary. There exist an $n = n_j$ and $\delta > 0$ such that $C \subset B(C_n, \delta)$ and $|f(x) - f(y)| < \varepsilon$ if $|x - y| < \delta + \frac{1}{n}$ ($x, y \in K$).

Let $y_i \in C_n$ be such that $|x_i - y_i| < \delta$ ($i = 1, \dots, k$) and $y_1 \leq y_2 \leq \dots \leq y_k$. By the definition of f_n , there exist $z_i \in K$ such that $f_n(y_i) = f(z_i)$ and $|z_i - y_i| \leq \frac{1}{n}$ ($i = 1, \dots, k$).

Since $|z_i - x_i| < \delta + \frac{1}{n}$, we have $|f(z_i) - f(x_i)| < \varepsilon$. Using that $\text{Var } f_n|_{C_n} \leq B$, we have

$$B \geq \sum_{i=1}^{k-1} |f_n(y_i) - f_n(y_{i+1})| = \sum_{i=1}^{k-1} |f(z_i) - f(z_{i+1})|,$$

and thus

$$\sum_{i=1}^{k-1} |f(x_i) - f(x_{i+1})| \leq B + 2k\varepsilon.$$

This holds for all $\varepsilon > 0$, therefore the total variation of $f|_C$ is at most B . \square

Note that the sets C_n in the previous lemmas are all contained in a compact interval, hence the sequence (C_n) has an accumulation point in the Hausdorff metric.

2.4.2 Proof of Theorem 2.1.1

Proposition 2.4.3. *Let $K \subset \mathbb{R}$ be a compact set of positive Lebesgue measure, let $f : K \rightarrow \mathbb{R}$ be continuous, and let $s < 1/2$, $B > 0$. There exists a compact set $C \subset K$ of Hausdorff dimension at least s such that $\text{Var } f|_C \leq B$.*

Proof. We may suppose without loss of generality that $f(K) \subset [0, 1]$. Let K_n and f_n be defined as above.

Recall that we denote the Lebesgue measure by λ . Let $\lambda_n = n \cdot \lambda(K_n)$; this is a positive integer. Then

$$K_n = \left[\frac{\varphi_n(0)}{n}, \frac{\varphi_n(0) + 1}{n} \right] \cup \dots \cup \left[\frac{\varphi_n(\lambda_n - 1)}{n}, \frac{\varphi_n(\lambda_n - 1) + 1}{n} \right]$$

for some integers $\varphi_n(0) < \varphi_n(1) < \dots < \varphi_n(\lambda_n - 1)$. Note that $\varphi_n : \lambda_n \rightarrow \mathbb{Z}$ is a non-contractive mapping. Define the function $g_n : \lambda_n \rightarrow [0, 1]$ by setting

$$g_n(k) = f_n\left(\frac{\varphi_n(k)}{n}\right) \quad (k \in \lambda_n). \quad (2.10)$$

Let us apply Theorem 2.3.2 to the functions g_n . We obtain some $\varepsilon > 0$ and subsets $A_n \subset \lambda_n$ such that $\mu^s(A_n) \geq \lambda_n^s \varepsilon \geq \lambda(K)^s n^s \varepsilon$ and $\text{Var } g_n|_{A_n} \leq B$.

Let $C_n = n^{-1} \cdot (\varphi_n(A_n))^*$ (see the definition of $*$ in Lemma 2.2.4), thus $C_n \subset K_n$. It is easy to see that we have $\text{Var } f_n|_{C_n} = \text{Var } g_n|_{A_n} \leq B$. From Lemma 2.2.4 we obtain

$$\mathcal{H}_\infty^s(C_n) = n^{-s} \mathcal{H}_\infty^s((\varphi_n(A_n))^*) \geq n^{-s} 2^{-s} \mu^s(\varphi_n(A_n)),$$

and since φ_n is a non-contractive mapping we get from Lemma 2.2.6 that

$$\mathcal{H}_\infty^s(C_n) \geq n^{-s} 2^{-s} \mu^s(A_n) \geq n^{-s} 2^{-s} \lambda(K)^s n^s \varepsilon = \lambda(K)^s 2^{-s} \varepsilon.$$

Now choose an accumulation point C of (C_n) . We immediately see from Lemma 2.4.1 that $C \subset K$, $\mathcal{H}_\infty^s(C) \geq \lambda(K)^s 2^{-s} \varepsilon > 0$, thus the Hausdorff dimension of C is at least s ; and from Lemma 2.4.2 we conclude that $\text{Var } f|_C \leq B$. \square

Now we are ready to prove our main theorem about bounded variation.

Proof of Theorem 2.1.1. There exists a compact set $K \subset [0, 1]$ of positive Lebesgue measure such that $f|_K$ is continuous. We may suppose that every non-empty intersection of K with an open interval has positive Lebesgue measure, since we may remove those non-empty intersections from K which are of Lebesgue measure zero (and we need to remove only countably many). Therefore we may use Proposition 2.4.3 not only for K , but for any non-empty portion of K .

Let $x \in K$, and let $x_n \searrow x$ be a strictly decreasing sequence in K converging fast enough to ensure

$$\sum_{n=1}^{\infty} \sup_{y \in [x, x_n] \cap K} |f(y) - f(x)| \leq 1.$$

For each positive integer n , let us apply Proposition 2.4.3 to the function f restricted to $K \cap [x_{2n+2}, x_{2n}]$ to obtain a compact set $C_n \subset K \cap [x_{2n+2}, x_{2n}]$ of dimension at least $1/2 - 1/n$ such that $\text{Var } f|_{C_n} \leq 2^{-n}$. Let C be the closure of $\bigcup_n C_n$ (which is $\bigcup_n C_n \cup \{x\}$). Thus C is of dimension at least $1/2$ and $\text{Var } f|_C \leq 1 + \sum_n 2^{-n} = 2$.

We may choose a compact subset of C of dimension exactly $1/2$ (see e.g. [10]), which concludes the proof. \square

2.5 Hölder restrictions

2.5.1 Overview of the proof

We give an informal outline of the proof of Theorem 2.1.2.

Let $1 \leq q < p$ be integers. A theorem of Mattila states that if $A \subset \mathbb{R}^p$ is an analytic set and $\mathcal{H}^s(A) > 0$, then we can find “many” q -dimensional planes $W \subset \mathbb{R}^p$ such that $\dim(A \cap W) \geq s + q - p$.

Let $m < n$ and let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be Borel measurable. Let $A \subset \mathbb{R}^n \times \mathbb{R}^m$ be the graph of f . Then clearly $\mathcal{H}^n(A) > 0$. Applying the previous theorem with $p = n + m$ and $q = n$, we obtain an affine mapping $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that $\dim(A \cap \text{graph } \varphi) \geq n + n - (n + m) = n - m$. This implies that f is actually linear (affine) restricted to a set of dimension $n - m$. Thus f is Lipschitz on a set of dimension $n - m$.

We can transform this result to the case of $\mathbb{R} \rightarrow \mathbb{R}$ functions. It is possible to map a “large portion” of \mathbb{R} to a “large portion” of \mathbb{R}^n by a Hölder- $1/n$ mapping such that its inverse is Hölder- n (see §2.5.2 and Claim 2.5.1). Now let us pretend that there is a Hölder- $1/n$ mapping $p_n : \mathbb{R} \rightarrow \mathbb{R}^n$ such that its inverse is Hölder- n , and that there is an analogous map $p_m : \mathbb{R} \rightarrow \mathbb{R}^m$. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be an arbitrary Borel function. Set $f = p_m \circ g \circ p_n^{-1}$, this is an $\mathbb{R}^n \rightarrow \mathbb{R}^m$ map. Apply the previous result to f . Since f is Lipschitz on some $C \subset \mathbb{R}^n$ of dimension $n - m$, the function $g = p_m^{-1} \circ f \circ p_n$ must be Hölder- m/n on the set $p_n^{-1}(C)$ of dimension $1 - m/n$.

A suitable approximation of any number $0 < \alpha < 1$ by fractions m/n would give Theorem 2.1.2.

2.5.2 Preliminaries

Let us recall some notions about self-similar sets. We say that a non-empty compact set $F \subset \mathbb{R}^n$ is a self-similar set satisfying the strong separation condition if there exist contractive similarities $\varphi_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ($i = 1, 2, \dots, k$), $k \geq 2$, such that

$$F = \varphi_1(F) \cup^* \varphi_2(F) \cup^* \dots \cup^* \varphi_k(F),$$

where \cup^* denotes disjoint union. If the similarity ratio of φ_i is denoted by r_i , then the Hausdorff dimension s of F is determined by the well-known equation (see e.g. [10])

$$\sum_{i=1}^k r_i^s = 1.$$

It is also known that $\mathcal{H}^s(F)$ is positive and finite. Setting $\Omega_k = \{1, 2, \dots, k\}^{\mathbb{N}}$ (the coding space), we can define the coding map $\pi : \Omega_k \rightarrow F$ by

$$\pi((i_0, i_1, \dots)) = \lim_{m \rightarrow \infty} \varphi_{i_0} \circ \varphi_{i_1} \circ \dots \circ \varphi_{i_m}(0).$$

Now let $0 < \varepsilon < 1$, and let $n \geq 1$ be an integer. Let r be such that

$$2^n r^{1-\varepsilon} = 1 \tag{2.11}$$

holds. Then $r < 1/2^n$. For $i = 1, \dots, 2^n$ we define similarities $\varphi_i : \mathbb{R} \rightarrow \mathbb{R}$ by setting

$$\varphi_i(x) = rx + \frac{1-r}{2^n-1}(i-1) \quad (i = 1, \dots, 2^n).$$

Then $\varphi_i([0, 1])$ are disjoint compact intervals in $[0, 1]$. Let $E_{n,\varepsilon}$ be the self-similar set generated by the similarities $\varphi_1, \varphi_2, \dots, \varphi_{2^n}$; that is,

$$E_{n,\varepsilon} = \varphi_1(E_{n,\varepsilon}) \cup^* \varphi_2(E_{n,\varepsilon}) \cup^* \dots \cup^* \varphi_{2^n}(E_{n,\varepsilon}) \subset [0, 1].$$

From equation (2.11) we conclude that the Hausdorff dimension of $E_{n,\varepsilon}$ is $1 - \varepsilon$.

We also define a self-similar set in \mathbb{R}^n . Taking a direct product of the self-similar set $E_{1,\varepsilon} \subset [0, 1]$, let

$$F_{n,\varepsilon} = \prod_{i=1}^n E_{1,\varepsilon} \subset [0, 1]^n$$

Then $F_{n,\varepsilon}$ is also self-similar, and its Hausdorff dimension is $n(1 - \varepsilon)$. Clearly $F_{n,\varepsilon}$ can be generated by 2^n similarities of scaling ratio $r^{1/n}$.

Claim 2.5.1. *There exists a bijection $p_n : E_{n,\varepsilon} \rightarrow F_{n,\varepsilon}$ such that p_n is Hölder- $1/n$ and its inverse is Hölder- n .*

Proof. The self-similar sets $E_{n,\varepsilon}$ and $F_{n,\varepsilon}$ have the same coding space Ω_{2^n} . It is easy to verify that $p_n = \pi_F \circ \pi_E^{-1}$ satisfies the requirements where π_F and π_E are the coding maps of $F_{n,\varepsilon}$ and $E_{n,\varepsilon}$, respectively. \square

Claim 2.5.2. *There exists a monotonic continuous surjective function $g : E_{n,\varepsilon} \rightarrow [0, 1]$ such that g is Hölder- $(1 - \varepsilon)$.*

Proof. We leave the proof to the reader. \square

2.5.3 Proofs

Recall that we say that $f \in B$ -Hölder $^\alpha$ (where $B > 0$) if $|f(x) - f(y)| \leq B|x - y|^\alpha$ for every x and y in the domain of f .

Proposition 2.5.3. *Let $K \subset [0, 1]$ be a compact set of positive Lebesgue measure, and let $f : K \rightarrow \mathbb{R}$ be continuous. Let $0 < \alpha < 1$, and $s < 1 - \alpha$. There exists a compact set $C \subset K$ of Hausdorff dimension at least s such that $f|_C \in 1$ -Hölder $^\alpha$.*

Proof. There exists a compact set $K' \subset K$ of positive Lebesgue measure such that $f(K') \subset [y, y + 1]$ for some $y \in \mathbb{R}$. Therefore we may suppose without loss of generality that $f(K) \subset [0, 1]$.

Choose positive integers $1 \leq m < n$ and some $0 < \varepsilon < 1$ such that

$$(1 - \varepsilon)m/n \geq \alpha \quad \text{and} \quad 1 - \varepsilon - m/n \geq s. \quad (2.12)$$

Let $c > 0$ be sufficiently small (depending on n , m , and ε).

Let μ be the Borel measure obtained by restricting $\mathcal{H}^{1-\varepsilon}$ to the self-similar set $E_{n,\varepsilon}$. Then $0 < \mu(\mathbb{R}) < \infty$. Let $G = \{(x, y) \in \mathbb{R}^2 : x + y \in K\}$. Since $\lambda(K) > 0$, an application of Fubini's theorem yields that

$$\int_{\mathbb{R}} \mu(K + t) \, d\lambda(t) = \lambda \times \mu(G) = \int_{\mathbb{R}} \lambda(K + t) \, d\mu(t) > 0.$$

Therefore there exists some $t \in \mathbb{R}$ such that $\mu(K + t) > 0$. We may suppose without loss of generality that in fact $t = 0$; that is, $\mu(K) > 0$. Then $\mathcal{H}^{1-\varepsilon}(K \cap E_{n,\varepsilon}) > 0$.

Let $g : E_{m,\varepsilon} \rightarrow [0, 1]$ be the function given by Claim 2.5.2. Let $h : [0, 1] \rightarrow E_{m,\varepsilon}$ be a Borel function such that $g \circ h$ is the identity. Then $h \circ f|_{K \cap E_{n,\varepsilon}} : K \cap E_{n,\varepsilon} \rightarrow E_{m,\varepsilon}$ is a Borel mapping. Since $\mathcal{H}^{1-\varepsilon}(K \cap E_{n,\varepsilon}) > 0$, by Luzin's theorem [12, Theorem 2.24]

there exists a compact set $D \subset K \cap E_{n,\varepsilon}$ of positive $\mathcal{H}^{1-\varepsilon}$ measure such that $h \circ f|_D$ is continuous.

Let $p_n : E_{n,\varepsilon} \rightarrow F_{n,\varepsilon}$ and $p_m : E_{m,\varepsilon} \rightarrow F_{m,\varepsilon}$ be the mappings given by Claim 2.5.1.

Let $D_n = p_n(D) \subset F_{n,\varepsilon}$; this is a compact set. We define a continuous function $\psi : D_n \rightarrow F_{m,\varepsilon} \subset \mathbb{R}^m$ by setting

$$\psi = p_m \circ h \circ f|_D \circ p_n^{-1}|_{D_n}. \quad (2.13)$$

Since the mapping p_n is Hölder- $1/n$ and its inverse is Hölder- n , from $\mathcal{H}^{1-\varepsilon}(D) > 0$ we obtain $\mathcal{H}^{n(1-\varepsilon)}(D_n) > 0$. Define the compact set

$$A = \text{graph } \psi = \{(x, \psi(x)) \in \mathbb{R}^{n+m} : x \in D_n\}.$$

As the projection of A to \mathbb{R}^n is D_n , we clearly have $\mathcal{H}^{n(1-\varepsilon)}(A) > 0$. A theorem of Mattila [10, Theorem 10.10] implies that for almost every n -plane in \mathbb{R}^{n+m} there exists a parallel n -plane W such that $A \cap W$ has Hausdorff dimension at least $n(1 - \varepsilon) + n - (n + m) = n(1 - \varepsilon) - m$. In fact, there exists an affine map $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with Lipschitz constant at most c such that $W = \text{graph } \varphi$ satisfies that $\dim A \cap W \geq n(1 - \varepsilon) - m$.

Let $C_n = \{x \in D_n : \psi(x) = \varphi(x)\}$ (a compact set). Since $A \cap \text{graph } \varphi$ is a Lipschitz (affine) image of C_n , we clearly have $\dim C_n \geq n(1 - \varepsilon) - m$. Clearly $\psi|_{C_n} = \varphi|_{C_n}$ is a Lipschitz function with constant at most c .

From (2.13) it is easy to deduce that

$$f|_D = g \circ (h \circ f|_D) = g \circ p_m^{-1} \circ \psi \circ p_n|_D.$$

Since p_n is Hölder- $1/n$, the map p_m^{-1} is Hölder- m , the function g is Hölder- $(1-\varepsilon)$, and $\psi|_{C_n}$ is Lipschitz, we obtain that $f|_D$ is a Hölder function with exponent $(1-\varepsilon)m/n$ on the compact set $C \stackrel{\text{def}}{=} p_n^{-1}(C_n) \subset D$. That is, $f|_C$ is a Hölder function with exponent $(1-\varepsilon)m/n$. Moreover, the Hölder constant for this exponent is clearly at most 1 if c is sufficiently small (compared to the Hölder constant of p_n, p_m^{-1} and g for the appropriate exponents). As $\dim C_n \geq n(1-\varepsilon) - m$, we have $\dim C \geq 1 - \varepsilon - m/n$. Therefore, using (2.12), we obtain that $f|_C \in 1\text{-Hölder}^\alpha$ and $\dim C \geq s$. \square

Proof of Theorem 2.1.2. There exists a compact set $K \subset [0, 1]$ of positive Lebesgue measure such that $f|_K$ is continuous. We may suppose that every non-empty intersection of K with an open interval has positive Lebesgue measure, since we may remove those non-empty intersections from K which are of Lebesgue measure zero (and we need to remove only countably many). Therefore we may use Proposition 2.5.3 not

only for K , but for any non-empty portion of K .

Let us apply Proposition 2.5.3 for some $0 < s < 1 - \alpha$ to obtain a compact set $C' \subset K$ of dimension at least s such that $f|_{C'} \in 1\text{-Hölder}^\alpha$. Choose a strictly decreasing sequence (x_n) in C' . Thus f is also Hölder- α with constant at most 1 on the sequence (x_n) . For each positive integer n , let $\varepsilon_n > 0$ be very small. Now for each positive integer n apply Proposition 2.5.3 to f restricted to $K \cap [x_n - \varepsilon_n, x_n + \varepsilon_n]$. We obtain compact sets $C_n \subset K \cap [x_n - \varepsilon_n, x_n + \varepsilon_n]$ of dimension at least $1 - \alpha - 1/n$ such that $f|_{C_n} \in 1\text{-Hölder}^\alpha$. Let C be the closure of $\bigcup_n C_n$. Thus C is of dimension at least $1 - \alpha$. It is clear that if the numbers ε_n are chosen to be small enough, then $C = \bigcup_n C_n \cup \{\lim x_n\}$, and from the continuity of f , that $f|_C \in 2\text{-Hölder}^\alpha$.

We may choose a compact subset of C of dimension exactly $1 - \alpha$, which concludes the proof. \square

2.6 Generalizations and open questions

Definition 2.6.1. The β -variation of a function $f : A \rightarrow \mathbb{R}$ (or $f : A \rightarrow \mathbb{R}^m$) is defined as

$$\sup \left\{ \sum_{i=1}^{n-1} |f(x_{i+1}) - f(x_i)|^\beta : x_1 < x_2 < \dots < x_n, x_i \in A \right\}.$$

Closely following the methods used in the proofs of Theorem 2.3.2 and Theorem 2.1.1, one can generalize these theorems to bounded β -variations instead of bounded 1-variation.

Theorem 2.6.2. *Let $f : [0, 1] \rightarrow \mathbb{R}$ be Lebesgue measurable, $\beta > 0$. There exists a compact set $C \subset [0, 1]$ of Hausdorff dimension $\frac{\beta}{1+\beta}$ such that f has finite β -variation on C .*

This result is sharp. Indeed, the methods of M. Elekes used in [4] can also be generalized to show that a typical continuous function has infinite β -variation on any set of dimension larger than $\frac{\beta}{1+\beta}$.

Using standard techniques it is straightforward to generalize Theorem 2.6.2 and Theorem 2.1.2 to higher dimensional Euclidean spaces. (Namely, we can exploit the fact that it is possible to map a “large portion” of \mathbb{R} to a “large portion” of \mathbb{R}^n by a Hölder- $1/n$ mapping such that its inverse is Hölder- n ; see Claim 2.5.1.)

Theorem 2.6.3. *Let $f : \mathbb{R} \rightarrow \mathbb{R}^m$ be Lebesgue measurable, $\beta > 0$. There exists a compact set $C \subset \mathbb{R}$ of Hausdorff dimension $\frac{\beta}{m+\beta}$ such that f has finite β -variation on C .*

Theorem 2.6.4. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be Lebesgue measurable and let $0 < \alpha < \frac{n}{m}$. There exists a compact set $C \subset \mathbb{R}^n$ of Hausdorff dimension $n - m\alpha$ such that f is Hölder- α on C .*

These theorems (that is, the stated dimensions) are again sharp for all β , m and n .

We have shown that every $\mathbb{R} \rightarrow \mathbb{R}$ Borel function is of bounded variation on some compact set of Hausdorff dimension $1/2$. However, we do not know anything about the possible ($1/2$ -dimensional) Hausdorff measure of such sets.

Question 4. Can we find for every Borel function $f : [0, 1] \rightarrow \mathbb{R}$ a Borel set B of positive $1/2$ -dimensional Hausdorff measure such that f restricted to B is of bounded variation?

A slightly more general variant is the following.

Question 5. Does there exist a Borel function $f : [0, 1] \rightarrow \mathbb{R}$ such that if f is of bounded variation on some Borel set B , then B has zero/finite/ σ -finite $1/2$ -dimensional Hausdorff measure?

The analogous questions for the Hölder- α property are also open.

Chapter 3

Borel maps and Hausdorff dimension

3.1 Outline

Our main aim in this chapter is to solve a problem of D. Preiss and B. Weiss (Question 1) as discussed in the Introduction (§1.3).

Theorem 3.1.1. *For every $0 \leq s < t \leq 1$ the measure spaces $(\mathbb{R}, \mathcal{B}, \mathcal{H}^s)$ and $(\mathbb{R}, \mathcal{B}, \mathcal{H}^t)$ are not isomorphic. Moreover, there does not exist a Borel bijection $f : \mathbb{R} \rightarrow \mathbb{R}$ such that for any Borel set $B \subset \mathbb{R}$,*

$$0 < \mathcal{H}^s(B) < \infty \iff 0 < \mathcal{H}^t(f(B)) < \infty. \quad (3.1)$$

Later we will also prove the analogous result in \mathbb{R}^n (see Theorem 3.7.5).

Theorem 3.1.2. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be Borel (or Lebesgue) measurable. For every $0 \leq d \leq 1$ there exists a compact set $A \subset \mathbb{R}$ such that $\dim A = d$ and $\dim f(A) \leq d$.*

Theorem 3.1.2 clearly implies Theorem 3.1.1. Indeed, let f be Borel measurable and choose a d for which $s < d < t$. By applying Theorem 3.1.2 we obtain a compact set A of dimension d with $\dim f(A) \leq d$. Since $s < d$, there exists a Borel subset B of A for which $0 < \mathcal{H}^s(B) < \infty$ (see e.g. [10]). Now $f(B) \subset f(A)$, so it has dimension at most d , which implies that $\mathcal{H}^t(f(B)) = 0$. So f cannot be an isomorphism of the measure spaces $(\mathbb{R}, \mathcal{B}, \mathcal{H}^s)$ and $(\mathbb{R}, \mathcal{B}, \mathcal{H}^t)$, and cannot satisfy (3.1) either.

To prove Theorem 3.1.2 it is clearly enough to show the following.

Theorem 3.1.3. *Suppose that K is a compact set of positive Lebesgue measure, and $f : K \rightarrow \mathbb{R}$ is continuous. For every $0 \leq d \leq 1$ there exists a compact set $A \subset K$ of Hausdorff dimension d such that $f(A)$ has Hausdorff dimension at most d .*

In fact, we will prove the following stronger theorem.

Theorem 3.1.4. *Suppose that K is a compact set of positive Lebesgue measure and $f : K \rightarrow \mathbb{R}$ is continuous. For every $0 \leq d \leq 1$ there exists a compact set $A \subset K$ of Hausdorff dimension d such that $f(A)$ has Hausdorff dimension at most $d \cdot \dim f(K)$.*

We outline the proof of this theorem. First, in §3.2 we prove some auxiliary lemmas. In §3.3 we define a random subtree of a specific rooted tree. Using this random subtree we construct random Cantor sets in §3.4 and prove an upper estimate for their dimension. In §3.5 we construct random compact sets and prove a lower estimate for their dimension using energy integrals. Then in §3.6 we prove Theorem 3.1.4 as follows: for the given K and continuous function $f : K \rightarrow \mathbb{R}$, we apply the former construction in the range space and the latter in the domain space in such a way that they produce a set A with the desired property.

Then, using straightforward techniques, we will deduce the following generalization of Theorem 3.1.2 from Theorem 3.1.4 in §3.7.

Theorem 3.1.5. *Let $D \subset \mathbb{R}^n$ be a Borel set and let $f : D \rightarrow \mathbb{R}^m$ be Borel measurable. Then for every $0 \leq d \leq 1$ there exists a Borel set $A \subset D$ such that $\dim A = d \cdot \dim D$ and $\dim f(A) \leq d \cdot \dim f(D)$.*

Remark 3.1.6. We can state Theorem 3.1.2 (almost) equivalently in the following form. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is Borel measurable and $0 \leq d \leq 1$, then there exists a compact set $B \subset \mathbb{R}$ such that $\dim B = d$ and $\dim f^{-1}(B) \geq d$. We might ask if there is a set $B \subset \mathbb{R}$ such that $\dim B = d$ and $\dim f^{-1}(B) \leq d$. However, this is far from true. In Chapter 4 we present a continuous real function f such that $f^{-1}(x)$ is of dimension 1 for every $x \in \mathbb{R}$, see Claim 4.5.20.

All results of this chapter were published in [8] with essentially the same proof. However, in [8] first we give an easier upper estimate (similar to §4.2) which is sufficient to prove Theorem 3.1.3 but not Theorem 3.1.4. We also have a strong upper estimate in [8] which is sufficient to prove Theorem 3.1.4 (and thus Theorem 3.1.5) as well. In fact, it turned out that this argument contained some (minor) error. Here we only give this stronger upper estimate correcting also the error made in [8].

3.2 Preliminaries

Notation. For a Borel measure μ on \mathbb{R} , let $I_t(\mu)$ denote the t -dimensional energy of μ ; that is, $I_t(\mu) = \iint |x - y|^{-t} d\mu(x) d\mu(y)$. For Borel measures μ_k ($k \in \mathbb{N}$) and μ , $\mu_k \rightarrow \mu$ denotes that μ_k weakly converges to μ .

The following statements are probably well-known.

Lemma 3.2.1. *Suppose that μ and μ_k ($k \in \mathbb{N}$) are probability measures on \mathbb{R} such that $\mu_k \rightarrow \mu$. Then $\mu_k \times \mu_k \rightarrow \mu \times \mu$.*

Proof. We have to show that for every compactly supported continuous function $h : \mathbb{R}^2 \rightarrow \mathbb{R}$, $\int_{\mathbb{R}^2} h d(\mu_k \times \mu_k) \rightarrow \int_{\mathbb{R}^2} h d(\mu \times \mu)$. Clearly it is enough to show this for a dense subset of the compactly supported continuous functions. It is well known that functions of the form

$$\sum_{i=1}^n f_i(x)g_i(y) \quad (f, g : \mathbb{R} \rightarrow \mathbb{R} \text{ continuous functions with compact support})$$

are dense, so it is enough to check that

$$\int_{\mathbb{R}^2} f(x)g(y) d(\mu_k \times \mu_k) \rightarrow \int_{\mathbb{R}^2} f(x)g(y) d(\mu \times \mu).$$

By Fubini,

$$\int_{\mathbb{R}^2} f(x)g(y) d(\mu_k \times \mu_k) = \int_{\mathbb{R}} f(x) d\mu_k(x) \int_{\mathbb{R}} g(y) d\mu_k(y)$$

which tends to

$$\int_{\mathbb{R}} f(x) d\mu(x) \int_{\mathbb{R}} g(y) d\mu(y) = \int_{\mathbb{R}^2} f(x)g(y) d(\mu \times \mu)$$

as $k \rightarrow \infty$, using $\mu_k \rightarrow \mu$ and Fubini again. \square

Lemma 3.2.2. *Suppose that μ_k ($k \in \mathbb{N}$) are probability measures on \mathbb{R} supported on $[-R, R]$ for some $R > 0$. If $\mu_k \rightarrow \mu$ then $I_t(\mu) \leq \liminf I_t(\mu_k)$.*

Proof. Let φ be a compactly supported continuous function on the plane which equals 1 on the square $[-R, R]^2$ and for which $0 \leq \varphi(x, y) \leq 1$ everywhere. For each positive integer i define $h_i : \mathbb{R}^2 \rightarrow \mathbb{R}$ by setting

$$h_i(x, y) = \varphi(x, y) \cdot \min(|x - y|^{-t}, i).$$

Using Lemma 3.2.1 we have

$$\begin{aligned} \int h_i(x, y) d\mu d\mu &= \lim_k \int h_i(x, y) d\mu_k d\mu_k \\ &\leq \liminf_{k \rightarrow \infty} \int |x - y|^{-t} d\mu_k d\mu_k = \liminf_{k \rightarrow \infty} I_t(\mu_k). \end{aligned}$$

The support of $\mu \times \mu$ is in $[-R, R]^2$ since μ_k is supported on $[-R, R]$ for all k , so we have

$$\lim_{i \rightarrow \infty} \int h_i(x, y) d\mu(x) d\mu(y) = \int |x - y|^{-t} d\mu(x) d\mu(y) = I_t(\mu).$$

Thus $I_t(\mu) \leq \liminf_{k \rightarrow \infty} I_t(\mu_k)$. □

Lemma 3.2.3. *Let $0 < t < 1$, let A be a compact set in \mathbb{R} and let $I = [0, \lambda(A)]$. Then*

$$\int_A \int_A |x - y|^{-t} d\lambda(x) d\lambda(y) \leq \int_I \int_I |x - y|^{-t} d\lambda(x) d\lambda(y) = c_t \lambda(A)^{2-t}$$

where c_t is a constant depending only on t .

Proof. Define the function $\varphi : A \rightarrow [0, \lambda(A)]$ by setting

$$\varphi(x) = \lambda((-\infty, x] \cap A).$$

Using first the fact that φ is 1-Lipschitz and then that it is a measure preserving transformation between $\lambda|_A$ and $\lambda|_I$, we obtain

$$\begin{aligned} \int_A \int_A |x - y|^{-t} d\lambda(x) d\lambda(y) &\leq \int_A \int_A |\varphi(x) - \varphi(y)|^{-t} d\lambda(x) d\lambda(y) \\ &= \int_I \int_I |x - y|^{-t} d\lambda(x) d\lambda(y) = \int_{[0,1]} \int_{[0,1]} |\lambda(A)x' - \lambda(A)y'|^{-t} \lambda(A)^2 d\lambda(x') d\lambda(y') \\ &= \lambda(A)^{2-t} \int_{[0,1]} \int_{[0,1]} |x' - y'|^{-t} d\lambda(x') d\lambda(y') = c_t \lambda(A)^{2-t} \end{aligned}$$

where c_t is finite if $t < 1$. □

3.3 The random tree

Let $M \geq 3$ and m be integers with $2 \leq m \leq M - 1$. Let

$$M^{<\omega} = \{(i_0, i_1, \dots, i_{n-1}) : n \in \mathbb{N}, i_\tau \in \{0, 1, \dots, M - 1\} = M\}.$$

We will consider $M^{<\omega}$ as a set of multi-indices and also as the M -adic tree with root \emptyset , where every node has M children. For an $i \in M^{<\omega}$ let $|i|$ denote the length of the multi-index; that is, the level of the node i .

For $i, j \in M^{<\omega}$ we write $i \leq j$ if j is a descendant of i in the tree. We say that i and j are incomparable if $i \not\leq j$ and $j \not\leq i$. If $i = (i_0, i_1, \dots, i_{n-1}) \in M^{<\omega}$ and $r \in M$, we adopt the notation $ir = (i_0, i_1, \dots, i_{n-1}, r)$.

We will use the notation $i \wedge j$ for the node which is the nearest common ancestor of i and j ; that is, $i \wedge j$ is the longest multi-index for which $i \wedge j \leq i$ and $i \wedge j \leq j$ hold.

Now we choose a “random m -adic subtree” S of $M^{<\omega}$ in the following way. Let X_i ($i \in M^{<\omega}$) be independent random variables with uniform distribution over the set of m -element subsets of M . That is, for each set $T \subset \{0, 1, \dots, M-1\}$ of m elements,

$$\mathbb{P}(X_i = T) = \frac{1}{\binom{M}{m}}.$$

Define the random subtree as

$$S = \{(i_0, i_1, \dots, i_{n-1}) \in M^{<\omega} : i_\tau \in X_{(i_0, i_1, \dots, i_{\tau-1})} \text{ for every } 0 \leq \tau \leq n-1\}.$$

So $\emptyset \in S$, and for each $i \in S$ exactly m children of i are in S . It is easy to see that

$$|\{i \in S : |i| = n\}| = m^n$$

for every $n \in \mathbb{N}$, and

$$\mathbb{P}(i \in S) = \left(\frac{m}{M}\right)^{|i|}. \quad (3.2)$$

Set

$$S_n = \{i \in S : |i| = n\} \quad (n \in \mathbb{N}).$$

3.4 Upper estimate

Let $E \subset \mathbb{R}$ be a compact set. Suppose that for each $i \in M^{<\omega}$ a compact non-trivial interval $U_i \subset \mathbb{R}$ is given satisfying the following conditions:

- (1) the endpoints of U_i lie in E ;
- (2) for every node i and its child ir ($r \in M$) we have $U_{ir} \subset U_i$; and
- (3) for every node i and for every two distinct $r, r' \in M$, the intervals U_{ir} and $U_{ir'}$ can have at most one point in common.

In fact (3) implies that if i and j are incomparable nodes, then U_i and U_j can have at most one point in common.

Using our random subtree $S \subset M^{<\omega}$ (see §3.3), we define random compact sets C_n as

$$C_n = \bigcup \{U_i \mid i \in S_n\}.$$

Then $U_\emptyset = C_0 \supset C_1 \supset C_2 \supset \dots$. Put

$$C = \bigcap_n C_n.$$

Lemma 3.4.1. *Almost surely $C \subset E$.*

Proof. It is easy to check that $\lim_{n \rightarrow \infty} \max_{i \in S_n} |U_i| = 0$ implies that $C \subset E$.

From the properties of the intervals U_i we know that

$$\sum_{|i|=n} |U_i| \leq \text{diam } E$$

for every n . Therefore for each fixed n , the number of those intervals U_i for which $|i| = n$ and $|U_i| \geq \delta$ is at most $(\text{diam } E)/\delta$. Hence the probability that there exists an $i \in S_n$ for which $|U_i| \geq \delta$ holds, tends to zero as $n \rightarrow \infty$. That is, the probability that $\max_{i \in S_n} |U_i| \geq \delta$ holds, tends to zero as $n \rightarrow \infty$. Keeping in mind that $\max_{i \in S_n} |U_i|$ is monotone decreasing, this implies that $\lim_{n \rightarrow \infty} \max_{i \in S_n} |U_i| \leq \delta$ almost surely. Since this holds for every $\delta > 0$, the proof is finished. \square

Proposition 3.4.2. *The random compact set C defined above has Hausdorff dimension at most $\frac{\log m}{\log M} \dim E$ almost surely.*

Proof. Let $t > \dim E$ be arbitrary and let $0 < \varepsilon < |U_\emptyset|^t$ be sufficiently small. Then $\mathcal{H}^t(E) = 0$ (this makes sense even if $t > 1$). Therefore we can choose a finite collection of open intervals \mathcal{I} covering the compact set E such that $\sum_{I \in \mathcal{I}} |I|^t < \varepsilon$. Clearly, we may suppose that \mathcal{I} is minimal in the sense that no interval is covered by the others. Then we can split \mathcal{I} into two collections \mathcal{I}' and \mathcal{I}'' such that both of them contains disjoint intervals only.

Let

$$\mathcal{I}_1 = \{I \in \mathcal{I} \mid \exists i \in M^{<\omega} U_i \subset I\}$$

and $\mathcal{I}_2 = \mathcal{I} \setminus \mathcal{I}_1$.

Let $I \in \mathcal{I}_1$ be arbitrary. Let $j(I)$ be a smallest multi-index for which $U_{j(I)} \subset I$ holds; that is, $U_i \not\subset I$ if $|i| < |j(I)|$. Since $\varepsilon < |U_\emptyset|^t$, we cannot have $U_\emptyset \subset I$, thus $|j(I)| \geq 1$.

If for three multi-indices i, i', i'' of length $|j(I)| - 1$ all the corresponding intervals $U_i, U_{i'}, U_{i''}$ intersected I , then at least one of them would be contained in I , which is impossible. Therefore at most two intervals U_i (with $|i| = |j(I)| - 1$) can intersect I , hence by (3.2)

$$\mathbb{P}(C \cap I \neq \emptyset) \leq 2 \left(\frac{m}{M}\right)^{|j(I)|-1}.$$

Now let $I \in \mathcal{I}_2$. Let N be a (large) positive integer. Then there can be at most two intervals U_i for which $|i| = N$ and $U_i \cap I \neq \emptyset$. Using (3.2) this implies that

$$\mathbb{P}(C \cap I \neq \emptyset) \leq 2 \left(\frac{m}{M} \right)^N.$$

Since this holds for all N , we obtain $\mathbb{P}(C \cap I \neq \emptyset) = 0$.

Since the intervals $I \in \mathcal{I}'$ are disjoint, the nodes $j(I)$ (when $I \in \mathcal{I}_1 \cap \mathcal{I}'$) form an anti-chain in $M^{<\omega}$; that is, none of them is an ancestor of any other. Thus

$$\sum_{I \in \mathcal{I}_1 \cap \mathcal{I}'} \frac{1}{M^{|j(I)|}} \leq 1.$$

The same inequality holds for the intervals in $\mathcal{I}_1 \cap \mathcal{I}''$, therefore

$$\sum_{I \in \mathcal{I}_1} \frac{1}{M^{|j(I)|}} \leq 2. \quad (3.3)$$

Let $s = \frac{\log m}{\log M} \cdot t$, hence $s < t$ and $m^{t/s} = M$.

By Lemma 3.4.1, $C \subset E$ almost surely. Therefore, almost surely, C can be covered by those intervals $I \in \mathcal{I}$ which intersect C . From the previous arguments we obtain

$$\begin{aligned} \mathbb{E}(\mathcal{H}_\infty^s(C)) &\leq \mathbb{E} \left(\sum_{I \in \mathcal{I}} |I|^s \mathbb{1}_{C \cap I \neq \emptyset} \right) \\ &= \sum_{I \in \mathcal{I}} \mathbb{P}(C \cap I \neq \emptyset) |I|^s \\ &= \sum_{I \in \mathcal{I}_1} \mathbb{P}(C \cap I \neq \emptyset) |I|^s \\ &\leq \sum_{I \in \mathcal{I}_1} 2 \left(\frac{m}{M} \right)^{|j(I)|} |I|^s \\ &\leq 2c \sum_{I \in \mathcal{I}_1} \frac{(m^{|j(I)|t/s} |I|^t)^{s/t}}{cM^{|j(I)|}}, \end{aligned}$$

where we choose c so that $\sum_{I \in \mathcal{I}_1} \frac{1}{cM^{|j(I)|}} = 1$ holds, hence $c \leq 2$ by (3.3). Applying Jensen's inequality to the concave function $x \mapsto x^{s/t}$ and using $m^{t/s} = M$ we obtain

$$\mathbb{E}(\mathcal{H}_\infty^s(C)) \leq 2c \sum_{I \in \mathcal{I}_1} \frac{(m^{|j(I)|t/s} |I|^t)^{s/t}}{cM^{|j(I)|}} \leq 2c \left(\sum_{I \in \mathcal{I}_1} \frac{m^{|j(I)|t/s} |I|^t}{cM^{|j(I)|}} \right)^{s/t}$$

$$\begin{aligned}
&= 2c \left(\sum_{I \in \mathcal{I}_1} \frac{|I|^t}{c} \right)^{s/t} = 2c^{1-s/t} \left(\sum_{I \in \mathcal{I}_1} |I|^t \right)^{s/t} \\
&\leq 4 \left(\sum_{I \in \mathcal{I}_1} |I|^t \right)^{s/t} \leq 4\varepsilon^{s/t}.
\end{aligned}$$

Because ε can be chosen arbitrarily small, we obtain $\mathbb{E}(\mathcal{H}_\infty^s(C)) = 0$ and thus $\mathcal{H}_\infty^s(C) = 0$ almost surely. Therefore $\dim C \leq s$ almost surely. This holds for every $s > \frac{\log m}{\log M} \dim E$, since $t > \dim E$ was arbitrary. So the dimension of C is at most $\frac{\log m}{\log M} \dim E$ almost surely. \square

3.5 Lower estimate

Suppose that for each $i \in M^{<\omega}$ a compact set $P_i \subset \mathbb{R}$ is given satisfying the following conditions:

- (1) $\lambda(P_i) = M^{-|i|}$;
- (2) for every node i and its child ir ($r \in M$) we have $P_{ir} \subset P_i$;
- (3) for every node i and for every two distinct $r, r' \in M$ the intersection of P_{ir} and $P_{ir'}$ is of Lebesgue measure zero.

In fact (3) implies that if i and j are incomparable then $\lambda(P_i \cap P_j) = 0$.

Using our random subtree $S \subset M^{<\omega}$ (see §3.3), we define random compact sets D_n as

$$D_n = \bigcup \{P_i \mid i \in S_n\}.$$

Then $P_\emptyset = D_0 \supset D_1 \supset D_2 \supset \dots$. Put

$$D = \bigcap_n D_n.$$

Theorem 3.5.1. *The random compact set D defined above has Hausdorff dimension at least $\frac{\log m}{\log M}$ almost surely.*

Proof. We define random Borel measures μ_k on \mathbb{R} by setting

$$\mu_k(A) = \lambda(A \cap D_k) \left(\frac{M}{m} \right)^k,$$

for every Borel set $A \subset \mathbb{R}$, or equivalently,

$$\mu_k = \left(\frac{M}{m}\right)^k \lambda|_{D_k} \quad (k \in \mathbb{N}). \quad (3.4)$$

Hence μ_k is a probability measure with support $D_k \subset P_\emptyset$.

Let $0 < t < \frac{\log m}{\log M}$ be fixed. We would like to give an upper bound for the expected value of the t -energy of μ_k . To do this at first we need to calculate some basic probability. We know that $\mathbb{P}(i \in S) = \left(\frac{m}{M}\right)^{|i|}$ for every $i \in M^{<\omega}$. How much is $\mathbb{P}(i \in S, j \in S)$ if $|i| = |j| = k$? Recall that $i \wedge j$ denotes the nearest common ancestor of i and j in the tree $M^{<\omega}$. Let $l = l(i, j) = |i \wedge j|$; that is, l is the largest integer for which $i_0 = j_0, i_1 = j_1, \dots, i_{l-1} = j_{l-1}$ hold ($0 \leq l \leq k$). Then

$$\begin{aligned} \mathbb{P}(i \in S, j \in S) &= \mathbb{P}\left(i_\tau \in X_{(i_0, \dots, i_{\tau-1})} \text{ for every } 0 \leq \tau \leq l-1, \right. \\ &\quad i_l, j_l \in X_{(i_0, \dots, i_{l-1})}, \\ &\quad i_\tau \in X_{(i_0, \dots, i_{\tau-1})} \text{ for every } l+1 \leq \tau \leq k-1, \\ &\quad \left. \text{and } j_\tau \in X_{(j_0, \dots, j_{\tau-1})} \text{ for every } l+1 \leq \tau \leq k-1\right). \end{aligned}$$

The random variables X_i are independent, so this probability is

$$\begin{aligned} &= \left(\frac{m}{M}\right)^l \frac{m(m-1)}{M(M-1)} \left(\frac{m}{M}\right)^{k-l-1} \left(\frac{m}{M}\right)^{k-l-1} \\ &= \left(\frac{m}{M}\right)^{2k-l-1} \frac{m-1}{M-1} \leq \left(\frac{m}{M}\right)^{2k-l} \end{aligned} \quad (3.5)$$

provided that $l < k$, that is, $i \neq j$, but the upper estimate clearly holds in the case $i = j$ ($l = k$) as well.

By (3.4), for any i of length k we have

$$\mu_k|_{P_i} = \begin{cases} \left(\frac{M}{m}\right)^k \lambda|_{P_i} & \text{if } i \in S \\ 0 & \text{if } i \notin S. \end{cases} \quad (3.6)$$

Applying first that $\text{supp } \mu_k = D_k$ is contained in $\bigcup_{|i|=k} P_i$, and then (3.6) and (3.5),

$$\begin{aligned} \mathbb{E} I_t(\mu_k) &= \mathbb{E} \left(\iint |x - y|^{-t} d\mu_k(x) d\mu_k(y) \right) \\ &= \mathbb{E} \left(\sum_{|i|=|j|=k} \int_{P_i} \int_{P_j} |x - y|^{-t} d\mu_k(x) d\mu_k(y) \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{|i|=|j|=k} \mathbb{E} \left(\int_{P_i} \int_{P_j} |x-y|^{-t} d\mu_k(x) d\mu_k(y) \right) \\
&= \sum_{|i|=|j|=k} \mathbb{P}(i \in S, j \in S) \int_{P_i} \int_{P_j} |x-y|^{-t} \left(\frac{M}{m}\right)^k \left(\frac{M}{m}\right)^k d\lambda(x) d\lambda(y) \\
&\leq \sum_{|i|=|j|=k} \left(\frac{m}{M}\right)^{2k-|i \wedge j|} \left(\frac{M}{m}\right)^{2k} \int_{P_i} \int_{P_j} |x-y|^{-t} d\lambda(x) d\lambda(y) \\
&= \sum_{|i|=|j|=k} \left(\frac{M}{m}\right)^{|i \wedge j|} \int_{P_i} \int_{P_j} |x-y|^{-t} d\lambda(x) d\lambda(y) \\
&= \sum_{l=0}^k \left(\frac{M}{m}\right)^l \sum_{\substack{h \\ |h|=l}} \sum_{\substack{i,j \\ h=i \wedge j \\ |i|=|j|=k}} \int_{P_i} \int_{P_j} |x-y|^{-t} d\lambda(x) d\lambda(y) \\
&\leq \sum_{l=0}^k \left(\frac{M}{m}\right)^l \sum_{\substack{h \\ |h|=l}} \sum_{\substack{i,j \\ h \leq i \wedge j \\ |i|=|j|=k}} \int_{P_i} \int_{P_j} |x-y|^{-t} d\lambda(x) d\lambda(y) \\
&= \sum_{l=0}^k \left(\frac{M}{m}\right)^l \sum_{\substack{h \\ |h|=l}} \int_{P_h} \int_{P_h} |x-y|^{-t} d\lambda(x) d\lambda(y). \tag{3.7}
\end{aligned}$$

Using Lemma 3.2.3 (and the constant c_t therein) we may continue (3.7) as

$$\begin{aligned}
&\leq \sum_{l=0}^k \left(\frac{M}{m}\right)^l \sum_{\substack{h \\ |h|=l}} c_t \lambda(P_h)^{2-t} \\
&= \sum_{l=0}^k \left(\frac{M}{m}\right)^l \sum_{\substack{h \\ |h|=l}} c_t \left(\frac{1}{M^l}\right)^{2-t} \\
&= \sum_{l=0}^k \left(\frac{M}{m}\right)^l M^l c_t \left(\frac{1}{M^l}\right)^{2-t} \\
&= \sum_{l=0}^k c_t \left(\frac{M^t}{m}\right)^l \leq \sum_{l=0}^{\infty} c_t \left(\frac{M^t}{m}\right)^l \stackrel{\text{def}}{=} c(t, M, m),
\end{aligned}$$

where $c(t, M, m)$ is finite whenever $\frac{M^t}{m} < 1$, that is, $t < \frac{\log m}{\log M}$.

By Fatou's lemma,

$$\mathbb{E} \liminf_{k \rightarrow \infty} I_t(\mu_k) \leq \liminf_{k \rightarrow \infty} \mathbb{E} I_t(\mu_k) \leq c(t, M, m),$$

thus $\liminf_{k \rightarrow \infty} I_t(\mu_k)$ is almost surely finite.

Since all the probability measures μ_k are supported on the same compact set P_\emptyset , every sequence of them has a weakly convergent subsequence. So we can choose a sequence of integers k_j such that

$$\lim_{j \rightarrow \infty} I_t(\mu_{k_j}) = \liminf_{k \rightarrow \infty} I_t(\mu_k)$$

and that μ_{k_j} is weakly convergent. Let $\mu = \lim_{j \rightarrow \infty} \mu_{k_j}$.

Since $\text{supp } \mu_{k_j} = D_{k_j}$ and $D_0 \supset D_1 \supset D_2 \supset \dots$, the weak limit μ is supported on $\bigcap_j D_{k_j} = D$. Applying Lemma 3.2.2,

$$I_t(\mu) \leq \liminf_{j \rightarrow \infty} I_t(\mu_{k_j}) = \liminf_{k \rightarrow \infty} I_t(\mu_k),$$

which is almost surely finite. Therefore the compact set D almost surely carries a measure μ with finite t -energy, for any $t < \frac{\log m}{\log M}$. Thus the Hausdorff dimension of the set D is at least $\frac{\log m}{\log M}$ almost surely. \square

Remark 3.5.2. It is possible to choose the sets P_i ($i \in M^{<\omega}$) in such a way that the $\frac{\log m}{\log M}$ dimensional Hausdorff measure of the random compact set D is zero almost surely.

3.6 Proof of Theorem 3.1.4

If there exists an $y \in f(K)$ for which $f^{-1}(y)$ is of positive measure, then we can choose a compact set $A \subset f^{-1}(y)$ of Hausdorff dimension d ($0 \leq d \leq 1$), and clearly $f(A) = \{y\}$ has Hausdorff dimension 0. Thus we may assume that for every $y \in f(K)$ the set $f^{-1}(y)$ has Lebesgue measure zero. Without loss of generality we may suppose that $\lambda(K) = 1$.

We will use the notation of §3.3. Put $E = f(K)$. For every $i \in M^{<\omega}$ we will define a compact non-trivial interval U_i with endpoints in E . Informally speaking, what we do is the following. We define U_\emptyset to be the smallest interval which contains E . If an interval is already defined, then its M subintervals (its children) are chosen such that their preimages (with respect to f) have equal Lebesgue measure: $\frac{1}{M}$ times the Lebesgue measure of the preimage of the interval. Now we give a more precise definition.

Define $\psi : f(K) \rightarrow \mathbb{R}$ as

$$\psi(y) = \lambda(\{x \in K : f(x) \leq y\}).$$

Since the preimage of any point in $f(K)$ has measure zero, ψ is a continuous increasing function, and its range is the interval $[0, \lambda(K)] = [0, 1]$.

For an $i \in M^{<\omega}$ we define U_i to be the compact interval with left endpoint

$$\max \left\{ y \in f(K) : \psi(y) = \sum_{\tau=1}^{|i|} \frac{i_{\tau-1}}{M^\tau} \right\}$$

and right endpoint

$$\min \left\{ y \in f(K) : \psi(y) = \frac{1}{M^{|i|}} + \sum_{\tau=1}^{|i|} \frac{i_{\tau-1}}{M^\tau} \right\}.$$

It is obvious from the definition that

$$U_i \supset \bigcup_{r \in M} U_{ir}$$

and

$$\lambda(f^{-1}(U_i)) = \lambda(\{x \in K : f(x) \in U_i\}) = \frac{1}{M^{|i|}}.$$

It is clear that the intervals U_i satisfy the assumptions of the upper estimate §3.4 with $E = f(K)$.

Define $P_i = f^{-1}(U_i)$ for every $i \in M^{<\omega}$. We know that $\lambda(P_i) = M^{-|i|}$. If i and j are incomparable, then U_i and U_j can have at most one point in common, thus $\lambda(P_i \cap P_j) = 0$. Therefore the compact sets P_i satisfy the assumptions of the lower estimate §3.5.

Now let S be the random m -adic subtree of $M^{<\omega}$, and define the random compact sets

$$C_n = \bigcup \{U_i : i \in S_n\} \quad (n \in \mathbb{N}),$$

$$C = \bigcap_{n \in \mathbb{N}} C_n,$$

and

$$D_n = \bigcup \{P_i : i \in S_n\} \quad (n \in \mathbb{N}),$$

$$D = \bigcap_{n \in \mathbb{N}} D_n,$$

the same way as in §3.4 and §3.5. Notice that $f^{-1}(C) = D$.

From Lemma 3.4.1 we know that $C \subset f(K)$ almost surely. From Proposition 3.4.2, $C = f(D)$ has Hausdorff dimension at most $\frac{\log m}{\log M} \dim f(K)$ almost surely. From Theorem 3.5.1 we know that D has Hausdorff dimension at least $\frac{\log m}{\log M}$ almost

surely. Therefore there exists a compact set $D \subset K$ for which both of the inequalities $\dim f(D) \leq \frac{\log m}{\log M} \dim f(K)$ and $\dim D \geq \frac{\log m}{\log M}$ hold.

For $d = 0$ or $d = 1$ the statement of the theorem is trivial, so let $0 < d < 1$ be arbitrary. Let

$$R = \left\{ \frac{\log m}{\log M} : 2 \leq m < M \right\},$$

this is a countable dense set in $(0, 1)$. We can construct compact sets D_r for every $r \in R$ such that D_r is of dimension at least r and $f(D_r)$ is of dimension at most $r \cdot \dim f(K)$. Let $D = \bigcup_{r < d} D_r$. Clearly D is a Borel set of dimension at least d , and $f(D) = \bigcup_{r < d} f(D_r)$ is of dimension at most $d \cdot \dim f(K)$. It is well known that D contains compact subsets A_n of dimension at least $d - 1/n$, and clearly we can require that A_n have diameter at most $1/n$. Let A be the closure of $\bigcup_n A_n$, then $A \setminus \bigcup_n A_n$ is at most one point. Thus $A \subset K$, $\dim A = d$, and clearly $\dim f(A) \leq d \cdot \dim f(K)$, which proves the theorem. \square

3.7 Generalization of Theorem 3.1.2 to Euclidean spaces

In this section we will prove Theorem 3.1.5. As a first step, observe that Theorem 3.1.4 implies the following.

Proposition 3.7.1. *Let $f : [0, 1] \rightarrow \mathbb{R}$ be a Borel function. For every $0 \leq d \leq 1$ there exists a compact set $A \subset [0, 1]$ such that $\dim A = d$ and $\dim f(A) \leq d \cdot \dim f([0, 1])$.* \square

Now we change the domain of the function f .

Proposition 3.7.2. *Let $D \subset \mathbb{R}$ be a Borel set and let $f : D \rightarrow \mathbb{R}$ be Borel. For every $0 \leq d \leq 1$ there exists a Borel set $A \subset D$ such that $\dim A = d \cdot \dim D$ and $\dim f(A) \leq d \cdot \dim f(D)$.*

Proof. The statement is trivial if $\dim D = 0$, therefore suppose that $\dim D > 0$.

Fix a positive number $s < \dim D$. It is well-known (see e.g. [10]) that there exists a compact set $D_s \subset D$ for which $\dim D_s > s$. Then there exist a positive constant c and a probability measure ν with $\text{supp } \nu \subset D_s$ such that for every $x, y \in \mathbb{R}$ we have

$$\nu([x, y]) \leq c |x - y|^s \tag{3.8}$$

(see again [10]). Let us define the continuous function $\psi : D_s \rightarrow [0, 1]$ and the Borel function $\chi : [0, 1] \rightarrow D_s$ by setting

$$\psi(x) = \nu((-\infty, x]),$$

$$\chi(y) = \min\{x : \psi(x) = y\}.$$

Thus $\psi \circ \chi$ is the identity of $[0, 1]$. The estimate (3.8) implies that ψ is a Hölder function of exponent s . Therefore for every set $A \subset [0, 1]$,

$$\dim \chi(A) \geq s \cdot \dim A. \quad (3.9)$$

Apply Proposition 3.7.1 to the Borel function $f \circ \chi : [0, 1] \rightarrow \mathbb{R}$. We get that for every $0 \leq d \leq 1$ there exists a compact set $A \subset [0, 1]$ such that

$$\dim A = d \quad \text{and} \quad \dim f(\chi(A)) \leq d \cdot \dim f(D_s) \leq d \cdot \dim f(D).$$

Put $B_s = \chi(A)$. (This is a Borel set, since χ is injective.) Applying (3.9) gives

$$\dim B_s \geq d \cdot s \quad \text{and} \quad \dim f(B_s) \leq d \cdot \dim f(D).$$

Now choose an increasing sequence (s_n) of positive numbers for which $s_n \rightarrow \dim D$. From the above procedure we obtain Borel sets $B_{s_n} \subset D$ satisfying

$$\dim B_{s_n} \geq d \cdot s_n \quad \text{and} \quad \dim f(B_{s_n}) \leq d \cdot \dim f(D).$$

Now any Borel subset of $\bigcup_n B_{s_n}$ of dimension $d \cdot \dim D$ is an appropriate choice for A . Thus the proof is finished. \square

The next lemma will be used in Chapter 4 as well.

Lemma 3.7.3. *For every positive integer n there exists a Borel set $E_n \subset [0, 1]$ and a Borel bijection $p_n : E_n \rightarrow [0, 1]^n \subset \mathbb{R}^n$ such that for every (Borel) set $A \subset E_n$ we have*

$$\lambda(p_n(A)) = \lambda(A) \quad \text{and} \quad \dim p_n(A) = n \cdot \dim A.$$

Moreover, for every $0 \leq s \leq 1$ and $A \subset E_n$,

$$c_{n,s} \mathcal{H}^s(A) \leq \mathcal{H}^{sn}(p_n(A)) \leq c_{n,s}^{-1} \mathcal{H}^s(A) \quad (3.10)$$

for some positive constant $c_{n,s}$ depending only on n and s .

Proof. For $x \in [0, 1)$ let $d_k(x) \in \{0, 1, \dots, 9\}$ ($k \geq 1$) denote the digits of x in the decimal number system; that is,

$$x = \sum_{k \geq 1} d_k(x) \cdot 10^{-k},$$

where $\liminf_{k \rightarrow \infty} d_k(x) \neq 9$. Let

$$E_n = \{x \in [0, 1) : \forall j \in \{0, 1, \dots, n-1\} \liminf_{i \rightarrow \infty} d_{ni-j}(x) \neq 9\},$$

$$p_n^j(x) = \sum_{i \geq 1} d_{ni-j}(x) \cdot 10^{-i} \quad (j \in \{0, 1, \dots, n-1\})$$

and

$$p_n(x) = (p_n^0(x), p_n^1(x), \dots, p_n^{n-1}(x)).$$

Then E_n is a Borel set of Lebesgue measure 1, and p_n is a Borel bijection between E_n and $[0, 1)^n$. It is easy to check that if $i \geq 1$ and $0 \leq k < 10^{ni}$ are integers, then p_n maps

$$E_n \cap \left[\frac{k}{10^{ni}}, \frac{k+1}{10^{ni}} \right)$$

to a cube of the form

$$\prod_{j=0}^{n-1} \left[\frac{k_j}{10^i}, \frac{k_j+1}{10^i} \right).$$

This implies that p_n preserves Lebesgue measure (that is, $\lambda(p_n(A)) = \lambda(A)$ for every measurable set $A \subset E_n$); and also implies that (3.10) is satisfied. Then $\dim p_n(A) = n \cdot \dim A$ also holds. As for the details, see [11, Theorem 49] and its proof. \square

It is easy to deduce the following statement from Lemma 3.7.3.

Lemma 3.7.4. *For every positive integer n there exists a Borel set $B_n \subset \mathbb{R}$ and a Borel bijection $p_n : B_n \rightarrow \mathbb{R}^n$ such that for every (Borel) set $A \subset B_n$ we have*

$$\lambda(p_n(A)) = \lambda(A) \quad \text{and} \quad \dim p_n(A) = n \cdot \dim A.$$

Moreover, for every $0 \leq s \leq 1$ and $A \subset B_n$,

$$c_{n,s} \mathcal{H}^s(A) \leq \mathcal{H}^{sn}(p_n(A)) \leq c_{n,s}^{-1} \mathcal{H}^s(A)$$

for some positive constant $c_{n,s}$ depending only on n and s . \square

Proof of Theorem 3.1.5. Suppose that $D \subset \mathbb{R}^n$ is a Borel set and $f : D \rightarrow \mathbb{R}^m$ is Borel measurable. Let $d \in [0, 1]$ be arbitrary. Let p_n and p_m be as in Lemma 3.7.4.

Apply Proposition 3.7.2 to the Borel set $p_n^{-1}(D) \subset \mathbb{R}$ and Borel mapping

$$p_m^{-1} \circ f \circ p_n|_{p_n^{-1}(D)} : p_n^{-1}(D) \rightarrow p_m^{-1}(f(D)).$$

We obtain a Borel set $A \subset p_n^{-1}(D)$ such that

$$\dim A = d \cdot \dim p_n^{-1}(D) \quad \text{and} \quad \dim p_m^{-1} \circ f \circ p_n(A) \leq d \cdot \dim p_m^{-1}(f(D)).$$

Using Lemma 3.7.4 four times we get that

$$\dim p_n(A) = d \cdot \dim(D) \quad \text{and} \quad \dim f(p_n(A)) \leq d \cdot \dim f(D)$$

hold for the Borel set $p_n(A) \subset D$. □

Let \mathcal{B}_n denote the σ -algebra of Borel subsets of \mathbb{R}^n . Lemma 3.7.4 implies that the generalization of Theorem 3.1.1 in \mathbb{R}^n holds.

Theorem 3.7.5. *For every $0 \leq s < t \leq n$ the measure spaces $(\mathbb{R}^n, \mathcal{B}_n, \mathcal{H}^s)$ and $(\mathbb{R}^n, \mathcal{B}_n, \mathcal{H}^t)$ are not isomorphic. Moreover, there does not exist a Borel bijection $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that for any Borel set $B \subset \mathbb{R}^n$,*

$$0 < \mathcal{H}^s(B) < \infty \iff 0 < \mathcal{H}^t(f(B)) < \infty. \quad \square$$

Chapter 4

Borel maps and Hausdorff measures

In this chapter we generalize our results proved in §3 from Hausdorff dimension to Hausdorff measures. We organize this chapter in a similar manner to the previous one. First we define a tree (a ‘larger’ tree than that in §3) and a random subtree in §4.1. Using this random subtree, we prove an upper estimate for the Hausdorff measure of certain kind of random Cantor sets in §4.2. Using the same random subtree, we prove a lower estimate for the Hausdorff pre-measures of certain kind of random sets in §4.3. However, the main part of this proof (and some tedious calculations) are postponed until §4.4. By combining these upper and lower estimates we prove our main results in §4.5.

We suggest reading of the sections of this chapter in the order they are presented here. However, since the sections are only loosely related, the reader may find other orders more convenient. For example, the reader may start with §4.5 which contains the main results. Those who like to start a proof at its *formal* beginning should definitely read §4.4 prior to §4.3.

4.1 The random tree

Let us fix some real number $0 < s < 1$. Let N_1, N_2, \dots be positive integers which tend to infinity sufficiently rapidly. Let

$$L_n = N_1 N_2 \cdots N_n \quad (n = 1, 2, \dots). \quad (4.1)$$

We will define an infinite rooted tree. Let V be the set of ‘words’ $v = (i_1, i_2, \dots, i_n)$ where $i_j \in \{0, 1, \dots, N_j - 1\}$ ($1 \leq j \leq n$), and $n \geq 0$. This n is called the length of

v , which will be denoted by $|v|$. Note that by $n = 0$ we mean that $\emptyset \in V$.

We consider the natural tree structure on V . (Both the tree and its vertex set will simply be denoted by V .) That is, \emptyset is the root, and vertices of the form $u = (i_1, i_2, \dots, i_{n-1})$ and $v = (i_1, i_2, \dots, i_n)$ (where $n \geq 1$) are connected. We adopt the short notation $v = ui_n$ in this case.

Let V_n denote the set of vertices on level n ; that is,

$$V_n = \{v \in V : |v| = n\} = \prod_{j=1}^n \{0, 1, \dots, N_j - 1\}.$$

Then $|V_n| = L_n$ by (4.1). Let us also define $N_0 = L_0 = 1$; then $|V_0| = L_0$ also holds.

For two vertices $u, v \in V$ we write $u \leq v$ if v is a descendant of u in the tree; that is, the unique (simple) path from the root to v contains u . We say that u and v are incomparable if $u \not\leq v$ and $v \not\leq u$.

We say that an edge is on level n if it connects vertices on level $n-1$ and level n .

Now we choose a random subtree S of this tree V in the following way. Independently for all edges in the tree, we choose each edge on level n with probability $p_n = N_n^{s-1}$ for every $n \geq 1$. Then let S be the set of vertices which are reachable from the root using the chosen edges only. Then $\emptyset \in S$ always, and for a vertex $v \in V_n$,

$$\mathbb{P}(v \in S) = N_1^{s-1} \dots N_n^{s-1} = L_n^{s-1}.$$

Set

$$S_n = \{v \in S_n : |v| = n\} \quad (n \in \mathbb{N}).$$

It is easy to check that $\mathbb{E}(|S_n|) = L_n L_n^{s-1} = L_n^s$.

As we said at the very beginning of this section, we suppose that the sequence (N_i) is increasing “sufficiently rapidly”. We do not need this property for the results of §4.2, but this is crucial for our theorems in §4.3, §4.4 and §4.5. It would be inconvenient to state here exactly how fast the sequence (N_i) should increase. Instead, we will make the appropriate assumptions throughout §4.3 and §4.4. All these assumptions can be written in the form of lower bounds $N_n \geq \Phi(N_1, N_2, \dots, N_{n-1})$; therefore they can be simultaneously satisfied.

Note that the number $0 < s < 1$ is considered fixed throughout §4.2, §4.3 and §4.4. The sequence (N_i) depends on s .

4.2 Upper estimate

Suppose that for each $v \in V$ a closed interval $U_v \subset [0, 1]$ is given satisfying the following conditions:

- (1) if $u \leq v$, then $U_v \subset U_u$;
- (2) if v and u are incomparable, then U_v and U_u have at most one point in common.

Then using the random subtree $S \subset V$ (cf. §4.1) we define random compact sets C_n as

$$C_n = \bigcup \{U_v : v \in S_n\}, \quad (4.2)$$

and let

$$C = \bigcap_{n=1}^{\infty} C_n. \quad (4.3)$$

Proposition 4.2.1. *For the s -dimensional Hausdorff measure of the random set C we have*

$$\mathbb{E} \mathcal{H}^s(C) \leq 1.$$

Lemma 4.2.2. *Almost surely $\lim_{n \rightarrow \infty} \max_{v \in S_n} |U_v| = 0$.*

Proof. For each fixed n , the number of those intervals U_v for which $v \in V_n$ and $|U_v| \geq \delta$ is at most $1/\delta$. Hence the probability that there exists a $v \in S_n$ for which $|U_v| \geq \delta$ tends to zero as $n \rightarrow \infty$. That is, the probability that $\max_{v \in S_n} |U_v| \geq \delta$ holds, tends to zero as $n \rightarrow \infty$. Keeping in mind that $\max_{v \in S_n} |U_v|$ is monotone decreasing, this implies that $\lim_{n \rightarrow \infty} \max_{v \in S_n} |U_v| \leq \delta$ almost surely. Since this holds for every $\delta > 0$, the proof is finished. \square

Proof of Proposition 4.2.1. We will cover C by the intervals U_v ($v \in S_n$). We know from Lemma 4.2.2 that $\lim_{n \rightarrow \infty} \max_{v \in S_n} |U_v| = 0$ almost surely. This implies that

$$\mathcal{H}^s(C) \leq \liminf_{n \rightarrow \infty} \sum_{v \in S_n} |U_v|^s$$

almost surely. Hence by Fatou's lemma

$$\mathbb{E} \mathcal{H}^s(C) \leq \mathbb{E} \liminf_{n \rightarrow \infty} \sum_{v \in S_n} |U_v|^s \leq \liminf_{n \rightarrow \infty} \mathbb{E} \sum_{v \in S_n} |U_v|^s. \quad (4.4)$$

Applying Jensen's inequality to the concave function $x \mapsto x^s$ and using that

$\sum_{v \in V_n} |U_v| \leq 1$, we obtain

$$\begin{aligned} \mathbb{E} \sum_{v \in S_n} |U_v|^s &= \mathbb{E} \sum_{v \in V_n} |U_v|^s \mathbb{1}_{v \in S_n} = \sum_{v \in V_n} |U_v|^s \mathbb{P}(v \in S_n) \\ &= L_n^{s-1} \sum_{v \in V_n} |U_v|^s \leq L_n^s \left(L_n^{-1} \sum_{v \in V_n} |U_v| \right)^s \leq 1. \end{aligned}$$

Combining this with (4.4) we obtain $\mathbb{E} \mathcal{H}^s(C) \leq 1$. \square

4.3 Lower estimate

Suppose that for each $v \in V$ a measurable set $M_v \subset [0, 1]$ is given satisfying the following conditions:

- (1) $\lambda(M_v) = L_n^{-1}$ for every $v \in V_n$, $n \geq 0$;
- (2) $M_u \subset M_v$ if $u \geq v$;
- (3) $\lambda(M_u \cap M_v) = 0$ if u and v are incomparable.

Using our random subtree $S \subset V$ (cf. §4.1), set

$$D_n = \bigcup \{M_u : u \in S_n\} \quad (n \in \mathbb{N}).$$

These random measurable sets satisfy

$$[0, 1] \supset D_0 \supset D_1 \supset D_2 \supset \dots \quad (4.5)$$

Clearly $\mathbb{E} \lambda(D_n) = \sum_{v \in V_n} \mathbb{P}(v \in S_n) \lambda(M_v) = L_n L_n^{s-1} L_n^{-1} = L_n^{s-1}$.

Theorem 4.3.1. *If the sets M_v and D_n are defined as above, then*

$$\mathbb{E} \mathcal{H}_\delta^s(D_n) \geq 1 - o_\delta(1),$$

where $o_\delta(1)$ is a quantity which tends to zero as $\delta \rightarrow 0$, but does not depend on n .

Remark 4.3.2. The constants involved in $o_\delta(1)$ are independent of the choice of the sets M_v ; but they do depend on the choice of V .

Remark 4.3.3. The condition that the sets M_v are contained in $[0, 1]$ is not necessary.

Theorem 4.3.4. *If the sets M_v and D_n are defined as above, moreover, the sets M_v are compact, then*

$$\mathbb{E} \mathcal{H}^s \left(\bigcap_n D_n \right) \geq 1.$$

This result should be contrasted with Remark 3.5.2. First we show that Theorem 4.3.1 implies Theorem 4.3.4.

Proof of Theorem 4.3.4. If the sets M_v are compact, then the sets D_n are also compact. Then it is easy to check that (4.5) implies

$$\mathcal{H}_\delta^s \left(\bigcap_n D_n \right) = \lim_{n \rightarrow \infty} \mathcal{H}_\delta^s(D_n)$$

for every $\delta > 0$. From Theorem 4.3.1 we obtain

$$\mathbb{E} \mathcal{H}_\delta^s \left(\bigcap_n D_n \right) \geq 1 - o_\delta(1).$$

As $\mathcal{H}^s(A) \geq \mathcal{H}_\delta^s(A)$ for every set A , we conclude the proof. \square

Proof of Theorem 4.3.1. When calculating Hausdorff measures (or pre-measures), we may require that the covering intervals belong to some suitably chosen class \mathcal{I} . In particular, let \mathcal{I} be the following class of closed subintervals of $[0, 1]$. Let \mathcal{I} contain only intervals of lengths

$$\frac{1}{2^r} \left(1 + \frac{i}{r} \right) \quad (i = 0, 1, \dots, r-1; \quad r \geq 10). \quad (4.6)$$

For each such length l , let \mathcal{I} contain (say) $\lfloor l^{-1} \log l^{-1} \rfloor$ many intervals of length l , which are ‘uniformly distributed’¹ in $[0, 1]$. (Here $\log l^{-1}$ could be anything which tends to infinity sufficiently slowly as $l \rightarrow 0$.) Then it is easy to check that for all $\varepsilon > 0$ there exists $\delta > 0$ such that if J is an interval in $[0, 1]$ of diameter at most δ , then there exists an $I \in \mathcal{I}$ for which $J \subset I$ and $|I|/|J| < 1 + \varepsilon$. For a set $A \subset [0, 1]$, define

$$\mathcal{H}_{\delta, \mathcal{I}}^s(A) = \inf \left\{ \sum_i |I_i|^s : A \subset \bigcup_i I_i, \quad |I_i| \leq \delta, \quad I_i \in \mathcal{I} \right\}.$$

Therefore for every $\varepsilon > 0$,

$$\mathcal{H}_{\delta, \mathcal{I}}^s(A) \leq (1 + \varepsilon)^s \mathcal{H}_\delta^s(A) \quad (4.7)$$

¹That is, the left endpoints form an arithmetic progression, the left endpoint of the first interval is 0, and the right endpoint of the last interval is 1.

if δ is sufficiently small.

For each n we define a Borel measure μ_n on the real line satisfying $\mu(\mathbb{R} \setminus D_n) = 0$. For a Borel set $A \subset \mathbb{R}$ let

$$\mu_n(A) = L_n^{1-s} \cdot \lambda(D_n \cap A),$$

that is,

$$\mu_n = L_n^{1-s} \cdot \lambda|_{D_n}.$$

Since $\mathbb{E} \lambda(D_n) = L_n^{s-1}$, we have

$$\mathbb{E} \mu_n([0, 1]) = 1. \quad (4.8)$$

A possible approach to prove Theorem 4.3.1 could be the following. If we knew that (almost surely) for every interval J the estimate $\mu_n(J) \leq (1 + \varepsilon)|J|^s$ holds (say), then clearly for every covering of D_n by intervals J_i we would have

$$\sum |J_i|^s \geq \sum (1 + \varepsilon)^{-1} \mu_n(J_i) \geq (1 + \varepsilon)^{-1} \mu_n(D_n)$$

and thus $\mathcal{H}_\infty^s(D_n) \geq (1 + \varepsilon)^{-1} \mu_n(D_n)$. By taking expected values, we would obtain $\mathbb{E} \mathcal{H}_\infty^s(D_n) \geq (1 + \varepsilon)^{-1}$. However, one can show that this approach is unable to yield Theorem 4.3.1 in its full strength. Instead, we will show that those intervals J for which $\mu_n(J) > (1 + \varepsilon)|J|^s$ holds cover only a small portion of μ_n (in expected value). More precisely we will show the following.

Lemma 4.3.5. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that for every n we have*

$$\sum_{\substack{I \in \mathcal{I} \\ |I| \leq \delta}} \mathbb{E} \mu_n(I) \mathbb{1}_{\mu_n(I) > (1+\varepsilon)|I|^s} < \varepsilon. \quad (4.9)$$

(In fact, δ does not even depend on the choice of the sets M_v .)

First let us finish the proof assuming Lemma 4.3.5. Let $\varepsilon > 0$ be arbitrary, and fix δ so that both (4.9) and (4.7) hold.

Consider a family of (distinct) intervals $J_i \in \mathcal{I}$ of diameters at most δ covering D_n . Then

$$\begin{aligned} \mu_n(D_n) &= \mu_n\left(\bigcup_i J_i\right) \leq \sum_i \mu_n(J_i) \\ &= \sum_i \mu_n(J_i) \mathbb{1}_{\mu_n(J_i) > (1+\varepsilon)|J_i|^s} + \sum_i \mu_n(J_i) \mathbb{1}_{\mu_n(J_i) \leq (1+\varepsilon)|J_i|^s} \end{aligned}$$

$$\begin{aligned}
&\leq \sum_i \mu_n(J_i) \mathbb{1}_{\mu_n(J_i) > (1+\varepsilon)|J_i|^s} + (1+\varepsilon) \sum_i |J_i|^s \\
&\leq \sum_{\substack{I \in \mathcal{I} \\ |I| \leq \delta}} \mu_n(I) \mathbb{1}_{\mu_n(I) > (1+\varepsilon)|I|^s} + (1+\varepsilon) \sum_i |J_i|^s. \tag{4.10}
\end{aligned}$$

Using the definition of $\mathcal{H}_{\delta, \mathcal{I}}^s(D_n)$, (4.10) implies that

$$(1+\varepsilon)\mathcal{H}_{\delta, \mathcal{I}}^s(D_n) \geq \mu_n(D_n) - \sum_{\substack{I \in \mathcal{I} \\ |I| \leq \delta}} \mu_n(I) \mathbb{1}_{\mu_n(I) > (1+\varepsilon)|I|^s}.$$

Combining this with (4.7) we obtain

$$(1+\varepsilon)^{1+s}\mathcal{H}_{\delta}^s(D_n) \geq \mu_n(D_n) - \sum_{\substack{I \in \mathcal{I} \\ |I| \leq \delta}} \mu_n(I) \mathbb{1}_{\mu_n(I) > (1+\varepsilon)|I|^s}.$$

Taking expected values and using (4.9) and (4.8) gives

$$(1+\varepsilon)^{1+s} \mathbb{E} \mathcal{H}_{\delta}^s(D_n) \geq 1 - \varepsilon.$$

From this one can deduce that $\mathbb{E} \mathcal{H}_{\delta}^s(D_n) \geq 1 - o_{\delta}(1)$, where $o_{\delta}(1)$ tends to zero as $\delta \rightarrow 0$ but does not depend on n . \square

It remains to prove Lemma 4.3.5. First we give an informal overview of the proof. Clearly it suffices to have a good upper estimate for $\mathbb{E}(\mu_n(I) \mathbb{1}_{\mu_n(I) > (1+\varepsilon)|I|^s})$ for each interval $I \in \mathcal{I}$. It is not difficult to show that

$$\frac{\mathbb{E} \mu_n(I)^2}{(1+\varepsilon)|I|^s}$$

is an upper estimate. This way the problem could be reduced to finding good upper bounds for the second moment $\mathbb{E} \mu_n(I)^2$. However, it would turn out that even the best possible estimates are too weak to imply Lemma 4.3.5. Instead, we will estimate $\mathbb{E}(\mu_n(I) \mathbb{1}_{\mu_n(I) > (1+\varepsilon)|I|^s})$ using the exponential moment $\mathbb{E} \exp(t\mu_n(I))$ (see Lemma 4.3.7). To estimate this exponential moment, in fact we need to estimate quantities like $\mathbb{E} \exp(t|B \cap S_n|)$, where $B \subset V_n$ is some set (which depends on I). The next section (§4.4) is entirely devoted to the upper bounds of these quantities. In this section we prove Lemma 4.3.5 using the results of §4.4.

First we need two more lemmas.

Lemma 4.3.6. *Let $c > 0$, $t \geq 1/c$, and $x \geq c$. Then*

$$x \leq \frac{e^{tx}}{e^{tc}}c.$$

Proof. As $x \geq c$, the right hand side is monotone increasing in t . Therefore we may suppose that $t = 1/c$. Then the inequality (after rearrangement) becomes $ex/c \leq e^{x/c}$, which holds. \square

Lemma 4.3.7. *Let X be a real random variable, and $c > 0$ a constant. Then*

$$\mathbb{E} X \mathbb{1}_{X \geq c} \leq \frac{\mathbb{E} e^{tX}}{e^{tc}}c$$

whenever $t \geq 1/c$. \square

Proof of Lemma 4.3.5. We will use the results of §4.4. However, it is enough to read Definition 4.4.1 and Corollary 4.4.12 to understand this proof.

First consider the case when $n = 0$. Since $M_\emptyset \subset [0, 1]$ has Lebesgue measure 1, and $D_0 = M_\emptyset$ always, we have $\mu_0 = \lambda|_{[0,1]}$. Then the left hand side of (4.9) is clearly zero. Therefore we may suppose that $n \geq 1$. (In fact, for our purposes, it would be enough to prove Lemma 4.3.5 for large integers n only.)

For each interval $I \in \mathcal{I}$ we will estimate the corresponding term in (4.9) using Lemma 4.3.7 and Corollary 4.4.12.

Fix an $I \in \mathcal{I}$. There exist a unique integer $m \geq 1$ and a unique real number $1 \leq k < N_m$ such that $|I| = kL_m^{-1}$. To I we associate a weight function $b : V_n \rightarrow [0, 1]$ by setting

$$b(u) = \lambda(M_u \cap I)L_n \quad (u \in V_n).$$

Then $b(V_n) = |I|L_n = kL_m^{-1}L_n$ and $\mu_n(I) = b(S_n)L_n^{-s}$ by the definition of μ_n . Using Lemma 4.3.7 we obtain

$$\begin{aligned} \mathbb{E} \mu_n(I) \mathbb{1}_{\mu_n(I) > e^{1/m}|I|^s} &\leq \frac{\mathbb{E} \exp(t\mu_n(I))}{\exp(te^{1/m}|I|^s)} e^{1/m}|I|^s \\ &= \frac{\mathbb{E} \exp(tb(S_n)L_n^{-s})}{\exp(te^{1/m}k^sL_m^{-s})} e^{1/m}k^sL_m^{-s} \end{aligned}$$

whenever $t \geq e^{-1/m}|I|^{-s}$, which holds if $t \geq |I|^{-s} = k^{-s}L_m^s$. Substituting tL_n^s into t we obtain

$$\mathbb{E} \mu_n(I) \mathbb{1}_{\mu_n(I) > e^{1/m}|I|^s} \leq \frac{\mathbb{E} \exp(tb(S_n))}{\exp(te^{1/m}k^sL_m^{-s}L_n^s)} e^{1/m}k^sL_m^{-s}$$

whenever $t \geq k^{-s} L_m^s L_n^{-s}$. Now we may use Corollary 4.4.12 and obtain

$$\mathbb{E} \mu_n(I) \mathbb{1}_{\mu_n(I) > e^{1/m} |I|^s} \leq \frac{k}{L_m} 3N_m^{-(1-s)/(8m)}.$$

We clearly have the same estimate for every interval in \mathcal{I} of the same length as I . Since there are exactly $\lfloor |I|^{-1} \log |I|^{-1} \rfloor = \lfloor k^{-1} L_m \log(k^{-1} L_m) \rfloor \leq k^{-1} L_m \log L_m$ many intervals in \mathcal{I} of length $|I|$, we obtain

$$\begin{aligned} \sum_{\substack{J \in \mathcal{I} \\ |J| = kL_m^{-1}}} \mathbb{E} \mu_n(J) \mathbb{1}_{\mu_n(J) > e^{1/m} |J|^s} &\leq \frac{k}{L_m} 3N_m^{-(1-s)/(8m)} k^{-1} L_m \log L_m \\ &= 3N_m^{-(1-s)/(8m)} \log L_m. \end{aligned} \quad (4.11)$$

From (4.6) it is easy to deduce that

$$\left| \{ |J| : L_m^{-1} \leq |J| < N_m L_m^{-1}, J \in \mathcal{I} \} \right| \leq 10(\log N_m)(\log L_m).$$

Combining this with (4.11),

$$\begin{aligned} \sum_{\substack{J \in \mathcal{I} \\ L_m^{-1} \leq |J| < N_m L_m^{-1}}} \mathbb{E} \mu_n(J) \mathbb{1}_{\mu_n(J) > e^{1/m} |J|^s} &\leq 30N_m^{-(1-s)/(8m)} (\log L_m)^2 (\log N_m) \\ &\leq 30N_m^{-(1-s)/(16m)}, \end{aligned}$$

where the last inequality holds if we suppose that (N_i) is increasing sufficiently rapidly.

Now let $\varepsilon > 0$ be arbitrary. Let m^* be a positive integer such that $e^{1/m^*} < 1 + \varepsilon$. Then

$$\begin{aligned} \sum_{\substack{J \in \mathcal{I} \\ |J| < N_{m^*} L_{m^*}^{-1}}} \mathbb{E} \mu_n(J) \mathbb{1}_{\mu_n(J) > (1+\varepsilon) |J|^s} &\leq \sum_{m=m^*}^{\infty} \sum_{\substack{J \in \mathcal{I} \\ L_m^{-1} \leq |J| < N_m L_m^{-1}}} \mathbb{E} \mu_n(J) \mathbb{1}_{\mu_n(J) > e^{1/m} |J|^s} \\ &\leq \sum_{m=m^*}^{\infty} 30N_m^{-(1-s)/(16m)} \\ &\leq 1/m^*, \end{aligned}$$

where the last inequality holds if (N_i) is increasing sufficiently rapidly. Since $1/m^* \leq e^{1/m^*} - 1 < \varepsilon$, we obtain (4.9) by choosing $\delta = \frac{1}{2} N_{m^*} L_{m^*}^{-1}$. \square

4.4 Estimates of the exponential moment

Before reading this section, we encourage the reader to consult §4.3 and especially the informal overview of the proof of Lemma 4.3.5 on page 44. We will use the notation and definitions of §4.1.

Definition 4.4.1. A weight function is a function $b : V_n \rightarrow [0, 1]$, where $n \geq 1$. For $A \subset V_n$, we write $b(A)$ for $\sum_{v \in A} b(v)$.

Definition 4.4.2. If $b : V_n \rightarrow [0, 1]$ is an arbitrary weight function, and $v \in V$ with $|v| \leq n$, then we define $b_v : V_n \rightarrow [0, 1]$ as

$$b_v(x) = \begin{cases} b(x) & \text{if } x \geq v, \\ 0 & \text{otherwise.} \end{cases}$$

We define the weight function $c_v : V_n \rightarrow \{0, 1\}$ as

$$c_v(x) = \begin{cases} 1 & \text{if } x \geq v, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for example, $c_v(V_n) = L_{|v|}^{-1} L_n$.

The theorems and proofs of this section are rather straightforward (though some involve a lot of calculations). The only exception is probably the following Lemma. (Finding this Lemma took the most time for the author when trying to prove Theorem 4.3.1 and thus the main results of this chapter.)

Lemma 4.4.3. *Let $b : V_n \rightarrow [0, 1]$ be a weight function. Let $0 \leq h \leq n$, $v \in V_h$. Then for every $t \geq 0$,*

$$\mathbb{E}(e^{tb_v(S_n)} \mid v \in S_h) \leq \left(\mathbb{E}(e^{tc_v(S_n)} \mid v \in S_h) \right)^{b_v(V_n)/c_v(V_n)}.$$

Proof. We prove the statement by backwards induction on h , starting from $h = n$.

If $h = n$, then the left hand side is $e^{tb_v(v)}$, while the right hand side is

$$(e^{tc_v(v)})^{b_v(V_n)/c_v(V_n)} = e^{tb_v(v)}$$

since $c_v(v) = c_v(V_n) = 1$.

Suppose now that $0 \leq h < n$ and the statement holds for $h + 1$. Let $i \in \{0, 1, \dots, N_{h+1} - 1\}$ and apply the induction hypothesis to the vertex $vi \in V_{h+1}$. We

obtain

$$\mathbb{E}(e^{tb_{vi}(S_n)} | v_i \in S_{h+1}) \leq \left(\mathbb{E}(e^{tc_{vi}(S_n)} | v_i \in S_{h+1}) \right)^{b_{vi}(V_n)/c_{vi}(V_n)}. \quad (4.12)$$

Since $0 \leq b_{vi}(V_n)/c_{vi}(V_n) \leq 1$, we may apply Jensen's inequality to the concave function $x \mapsto x^{b_{vi}(V_n)/c_{vi}(V_n)}$ so that it gives

$$(1-p)1 + py^{b_{vi}(V_n)/c_{vi}(V_n)} \leq ((1-p)1 + py)^{b_{vi}(V_n)/c_{vi}(V_n)} \quad (4.13)$$

for arbitrary $0 < p < 1$ and $y > 0$. Substituting $p = N_{h+1}^{s-1}$ and $y = \mathbb{E}(e^{tc_{vi}(S_n)} | v_i \in S_{h+1})$ into (4.13) and then applying (4.12) gives

$$\begin{aligned} 1 - N_{h+1}^{s-1} + N_{h+1}^{s-1} \mathbb{E}(e^{tb_{vi}(S_n)} | v_i \in S_{h+1}) \\ \leq \left(1 - N_{h+1}^{s-1} + N_{h+1}^{s-1} \mathbb{E}(e^{tc_{vi}(S_n)} | v_i \in S_{h+1}) \right)^{b_{vi}(V_n)/c_{vi}(V_n)}. \end{aligned}$$

Here the left hand side equals $\mathbb{E}(e^{tb_{vi}(S_n)} | v \in S_h)$, and we can rewrite the right hand side analogously to obtain

$$\mathbb{E}(e^{tb_{vi}(S_n)} | v \in S_h) \leq \left(\mathbb{E}(e^{tc_{vi}(S_n)} | v \in S_h) \right)^{b_{vi}(V_n)/c_{vi}(V_n)}. \quad (4.14)$$

Conditional on $v \in S_h$, the random variables $c_{vi}(S_n)$ are independent and they have identical distribution, and $\sum_i c_{vi} = c_v$. Therefore

$$\begin{aligned} \mathbb{E}(e^{tc_v(S_n)} | v \in S_h) &= \prod_{i=0}^{N_{h+1}-1} \mathbb{E}(e^{tc_{vi}(S_n)} | v \in S_h) \\ &= \left(\mathbb{E}(e^{tc_{vj}(S_n)} | v \in S_h) \right)^{N_{h+1}} \end{aligned}$$

for every $j \in \{0, 1, \dots, N_{h+1} - 1\}$. This implies that (4.14) is equivalent to

$$\mathbb{E}(e^{tb_{vi}(S_n)} | v \in S_h) \leq \left(\mathbb{E}(e^{tc_v(S_n)} | v \in S_h) \right)^{b_{vi}(V_n)/c_v(V_n)}. \quad (4.15)$$

Conditional on $v \in S_h$, the random variables $b_{vi}(S_n)$ are independent, and $b_v = \sum_i b_{vi}$. Therefore we have

$$\prod_{i=0}^{N_{h+1}-1} \mathbb{E}(e^{tb_{vi}(S_n)} | v \in S_h) = \mathbb{E}(e^{tb_v(S_n)} | v \in S_h).$$

Thus by multiplying (4.15) for all $i \in \{0, 1, \dots, N_{h+1} - 1\}$ we obtain

$$\mathbb{E}(e^{tb_v(S_n)} | v \in S_h) \leq \left(\mathbb{E}(e^{tc_v(S_n)} | v \in S_h) \right)^{b_v(V_n)/c_v(V_n)},$$

which is exactly what we wanted to prove. \square

Lemma 4.4.4. *Let $b : V_n \rightarrow [0, 1]$ be a weight function, and let $1 \leq m \leq n$. Then for every $t \geq 0$,*

$$\mathbb{E} e^{tb(S_n)} \leq 1 - L_{m-1}^{s-1} + L_{m-1}^{s-1} \prod_{v \in V_{m-1}} \mathbb{E}(e^{tb_v(S_n)} | v \in S_{m-1}).$$

Proof. We choose another random subtree of V in the following way. For each level $1 \leq h \leq m-1$, instead of choosing each edge with probability N_h^{s-1} independently of each other, we either choose all edges or choose none of them with probability N_h^{s-1} and $1 - N_h^{s-1}$, respectively. We do not modify anything else in the construction; that is, the edges on levels $\geq m$ are still chosen independently, and all events are considered as independent unless otherwise stated. This way we obtain the random sets $R_h \subset V_h$ of reachable vertices on level h ($h \geq 0$). Then for every positive integer τ clearly

$$\begin{aligned} \mathbb{E} b(S_n)^\tau &= \sum_{i_1, \dots, i_\tau \in V_n} b(i_1) \dots b(i_\tau) \mathbb{P}(i_1, \dots, i_\tau \in S_n) \\ &\leq \sum_{i_1, \dots, i_\tau \in V_n} b(i_1) \dots b(i_\tau) \mathbb{P}(i_1, \dots, i_\tau \in R_n) = \mathbb{E} b(R_n)^\tau. \end{aligned}$$

This immediately gives

$$\mathbb{E} e^{tb(S_n)} \leq \mathbb{E} e^{tb(R_n)} \tag{4.16}$$

for $t \geq 0$. Clearly

$$\mathbb{E} e^{tb(R_n)} = 1 - L_{m-1}^{s-1} + L_{m-1}^{s-1} \mathbb{E}(e^{tb(R_n)} | R_{m-1} = V_{m-1}). \tag{4.17}$$

Since $b(R_n) = \sum_{v \in V_{m-1}} b_v(R_n)$, we have by independence

$$\mathbb{E}(e^{tb(R_n)} | R_{m-1} = V_{m-1}) = \prod_{v \in V_{m-1}} \mathbb{E}(e^{tb_v(R_n)} | R_{m-1} = V_{m-1}). \tag{4.18}$$

Since $\mathbb{E}(e^{tb_v(R_n)} | R_{m-1} = V_{m-1}) = \mathbb{E}(e^{tb_v(S_n)} | v \in S_{m-1})$ for every $v \in V_{m-1}$, combining (4.16), (4.17) and (4.18) concludes the proof. \square

Corollary 4.4.5. *Let $b : V_n \rightarrow [0, 1]$ be a weight function, let $1 \leq m \leq n$, and $u \in V_{m-1}$. Then for every $t \geq 0$,*

$$\mathbb{E} e^{tb(S_n)} \leq 1 - L_{m-1}^{s-1} + L_{m-1}^{s-1} \left(\mathbb{E} \left(e^{tc_u(S_n)} \mid u \in S_{m-1} \right) \right)^{b(V_n)/c_u(V_n)}.$$

Proof. Combining Lemma 4.4.4 and Lemma 4.4.3 (for $h = m - 1$) gives

$$\mathbb{E} e^{tb(S_n)} \leq 1 - L_{m-1}^{s-1} + L_{m-1}^{s-1} \prod_{v \in V_{m-1}} \left(\mathbb{E} \left(e^{tc_v(S_n)} \mid v \in S_{m-1} \right) \right)^{b_v(V_n)/c_v(V_n)}.$$

Since $\mathbb{E} \left(e^{tc_v(S_n)} \mid v \in S_{m-1} \right) = \mathbb{E} \left(e^{tc_u(S_n)} \mid u \in S_{m-1} \right)$ for every $v \in V_{m-1}$, and $b(V_n) = \sum_{v \in V_{m-1}} b_v(V_n)$, we obtain the corollary. \square

Remark 4.4.6. In some sense, Corollary 4.4.5 and the following Lemma 4.4.7 can be considered as an analogue of Lemma 3.2.3 of Chapter 3.

Lemma 4.4.7. *Let $1 \leq m \leq n$, $u \in V_{m-1}$, and let c_u be the weight function as in Definition 4.4.2. Then*

$$\mathbb{E} \left(e^{tc_u(S_n)} \mid u \in S_{m-1} \right) \leq \exp \left(t L_{m-1}^{-s} L_n^s \exp(2t L_{m-1}^{-s} L_n^s) \right)$$

if $0 \leq t \leq L_m^s L_n^{-s}$; and

$$\mathbb{E} \left(e^{tc_u(S_n)} \mid u \in S_{m-1} \right) \leq \left(1 - N_m^{s-1} + N_m^{s-1} \exp \left(t L_m^{-s} L_n^s \exp(2t L_{m+1}^{-s} L_n^s) \right) \right)^{N_m}$$

if $0 \leq t \leq L_{m+1}^s L_n^{-s}$ and $1 \leq m < n$.

Proof. For $m \leq h \leq n + 1$, set

$$a_h = \mathbb{E} \left(e^{tc_v(S_n)} \mid v \in S_{h-1} \right)$$

where v is an arbitrary vertex in V_{h-1} . Using this notation, the first part of the Lemma states that $a_m \leq \exp \left(t L_{m-1}^{-s} L_n^s \exp(2t L_m^{-s} L_n^s) \right)$ if $0 \leq t \leq L_m^s L_n^{-s}$.

Since $c_v = \sum_i c_{vi}$, clearly

$$\begin{aligned} a_h &= \prod_{i=0}^{N_h-1} \mathbb{E} \left(e^{tc_{vi}(S_n)} \mid v \in S_{h-1} \right) \\ &= \left(\mathbb{E} \left(e^{tc_{v0}(S_n)} \mid v \in S_{h-1} \right) \right)^{N_h} \\ &= \left(1 - N_h^{s-1} + N_h^{s-1} \mathbb{E} \left(e^{tc_{v0}(S_n)} \mid v0 \in S_h \right) \right)^{N_h} \end{aligned}$$

$$= \left(1 + N_h^{s-1}(a_{h+1} - 1)\right)^{N_h} \quad (4.19)$$

$$\leq \exp(N_h^s(a_{h+1} - 1)) \quad (4.20)$$

if $m \leq h \leq n$.

We claim that the second part of the Lemma can be easily deduced from the first part. Indeed, suppose that $1 \leq m < n$. Then (4.19) implies that $a_m \leq (1 - N_m^{s-1} + N_m^{s-1}a_{m+1})^{N_m}$. Combine this inequality with the upper estimate for a_{m+1} given by the first part of the Lemma for $m+1$ in place of m . This gives exactly the second part of the Lemma with the right condition for t .

To prove the first part of the Lemma, suppose that $0 \leq t \leq L_m^s L_n^{-s}$. We have to show that $a_m \leq \exp(tL_{m-1}^{-s}L_n^s \exp(2tL_m^{-s}L_n^s))$. Clearly

$$a_{n+1} = e^t. \quad (4.21)$$

It is easy to check that

$$e^x - 1 \leq x(1 + x) \quad (4.22)$$

for every $0 \leq x \leq 1.7$. Therefore by (4.20),

$$a_n \leq \exp(N_n^s(e^t - 1)) \leq \exp(N_n^s t(1 + t)), \quad (4.23)$$

since clearly $t \leq L_m^s L_n^{-s} \leq 1 < 1.7$ as $m \leq n$. We claim that in general

$$a_h \leq \exp\left(tL_{h-1}^{-s}L_n^s \prod_{j=h}^n (1 + tL_j^{-s}L_n^s)^{2^{j-h}}\right) \quad (4.24)$$

for every $m \leq h \leq n+1$. First we prove the following.

Claim 4.4.8. *Suppose that $m \leq h \leq n$. Then*

$$A_h \stackrel{\text{def}}{=} tL_{h-1}^{-s}L_n^s \prod_{j=h}^n (1 + tL_j^{-s}L_n^s)^{2^{j-h}} \leq tL_{h-1}^{-s}L_n^s \exp(2tL_h^{-s}L_n^s).$$

Moreover, if $h \geq m+1$, then $A_h \leq 1.7$ (provided that $t \leq L_m^s L_n^{-s}$).

Proof. We may choose the integers N_i so large that $2N_i^{-s} \leq 1/2$ holds for every i . Clearly

$$A_h \leq tL_{h-1}^{-s}L_n^s \exp\left(\sum_{j=h}^n tL_j^{-s}L_n^s 2^{j-h}\right).$$

Since the sequence (N_i) is monotone increasing, for $j \geq h$ we clearly have $L_j \geq$

$L_h N_{h+1}^{j-h}$. This implies that $L_j^{-s} 2^{j-h} \leq L_h^{-s} (2N_{h+1}^{-s})^{j-h}$ for $j \geq h$. Therefore

$$A_h \leq tL_{h-1}^{-s} L_n^s \exp\left(tL_h^{-s} L_n^s \sum_{j=h}^{\infty} (2N_{h+1}^{-s})^{j-h}\right) \leq tL_{h-1}^{-s} L_n^s \exp(2tL_h^{-s} L_n^s),$$

where we used that $2N_{h+1}^{-s} \leq 1/2$.

If, in addition, $h \geq m+1$ and $t \leq L_m^s L_n^{-s}$ hold, then from the previous arguments we clearly have

$$A_h \leq tL_m^{-s} L_n^s \exp(2tL_{m+1}^{-s} L_n^s) \leq \exp(2N_{m+1}^{-s}) \leq \exp(1/2) \leq 1.7. \quad \square$$

We would like to prove (4.24). Notice that for $h = n+1$ and $h = n$ we are already done by (4.21) and (4.23). Suppose that $m \leq h \leq n-1$ and that (4.24) holds for $h+1$. Then combining this induction hypothesis with (4.22),

$$\begin{aligned} a_{h+1} - 1 &\leq \exp(A_{h+1}) - 1 \leq A_{h+1}(1 + A_{h+1}) \\ &= \left(tL_h^{-s} L_n^s \prod_{j=h+1}^n (1 + tL_j^{-s} L_n^s)^{2^{j-h-1}}\right) \left(1 + tL_h^{-s} L_n^s \prod_{j=h+1}^n (1 + tL_j^{-s} L_n^s)^{2^{j-h-1}}\right) \\ &\leq tL_h^{-s} L_n^s \left(\prod_{j=h+1}^n (1 + tL_j^{-s} L_n^s)^{2^{j-h-1}}\right) \left(\prod_{j=h+1}^n (1 + tL_j^{-s} L_n^s)^{2^{j-h-1}}\right) (1 + tL_h^{-s} L_n^s) \\ &= tL_h^{-s} L_n^s \prod_{j=h}^n (1 + tL_j^{-s} L_n^s)^{2^{j-h}}. \end{aligned}$$

Note that here (the second part of) Claim 4.4.8 grants that $A_{h+1} \leq 1.7$, which means that (4.22) is satisfied. Thus from (4.20),

$$a_h \leq \exp(N_h^s (a_{h+1} - 1)) \leq \exp\left(N_h^s tL_h^{-s} L_n^s \prod_{j=h}^n (1 + tL_j^{-s} L_n^s)^{2^{j-h}}\right),$$

which (noticing that $N_h^s L_h^{-s} = L_{h-1}^{-s}$) proves (4.24).

Combining (4.24) and Claim 4.4.8 for $h = m$ we obtain

$$a_m \leq \exp(tL_{m-1}^{-s} L_n^s \exp(2tL_m^{-s} L_n^s)),$$

which was to be proved. \square

Lemma 4.4.9. *Let $1 \leq m \leq n$ and let $b : V_n \rightarrow [0, 1]$ be a weight function with $b(V_n) = kL_m^{-1} L_n$ for some real number $1 \leq k \leq \sqrt{\log N_m}$. Then there exists some*

$t \geq k^{-s} L_m^s L_n^{-s}$ such that

$$\frac{\mathbb{E} e^{tb(S_n)}}{\exp(te^{1/m} k^s L_m^{-s} L_n^s)} e^{1/m} k^{s-1} L_m^{1-s} \leq 3N_m^{-(1-s)/(8m)}. \quad (4.25)$$

Lemma 4.4.10. *Let $1 \leq m \leq n$ and let $b : V_n \rightarrow [0, 1]$ be a weight function with $b(V_n) = kL_m^{-1}L_n$ for some real number $\sqrt{\log N_m} \leq k \leq N_m^{(1-s)/2}$. Then there exists some $t \geq k^{-s} L_m^s L_n^{-s}$ such that*

$$\frac{\mathbb{E} e^{tb(S_n)}}{\exp(te^{1/m} k^s L_m^{-s} L_n^s)} e^{1/m} k^{s-1} L_m^{1-s} \leq 8N_m^{-(\log N_m)^{s/4} + 1}. \quad (4.26)$$

Lemma 4.4.11. *Let $1 \leq m \leq n$ and let $b : V_n \rightarrow [0, 1]$ be a weight function with $b(V_n) = kL_m^{-1}L_n$ for some real number $N_m^{(1-s)/2} \leq k \leq N_m$. Then there exists some $t \geq k^{-s} L_m^s L_n^{-s}$ such that*

$$\frac{\mathbb{E} e^{tb(S_n)}}{\exp(te^{1/m} k^s L_m^{-s} L_n^s)} e^{1/m} k^{s-1} L_m^{1-s} \leq 3 \exp\left(-\frac{N_m^{s(1-s)/2}}{8m^2}\right) N_m. \quad (4.27)$$

Corollary 4.4.12. *Let $n \geq 1$, $m \geq 1$. Let $b : V_n \rightarrow [0, 1]$ be a weight function with $b(V_n) = kL_m^{-1}L_n$ for some real number $1 \leq k \leq N_m$. Then there exists some $t \geq k^{-s} L_m^s L_n^{-s}$ such that*

$$\frac{\mathbb{E} e^{tb(S_n)}}{\exp(te^{1/m} k^s L_m^{-s} L_n^s)} e^{1/m} k^{s-1} L_m^{1-s} \leq 3N_m^{-(1-s)/(8m)}. \quad (4.28)$$

Proof. First assume that $m \leq n$. Then (4.28) is a clear consequence of Lemmas 4.4.9, 4.4.10, 4.4.11 provided that we suppose that (N_i) is increasing sufficiently rapidly.

Now assume that $m \geq n + 1$. Since $b(S_n) \leq b(V_n)$, we have $\mathbb{E}(\exp(tb(S_n))) \leq \exp(tkL_m^{-1}L_n)$. Therefore the quotient on the left hand side of (4.28) is at most

$$\exp(t(kL_m^{-1}L_n - e^{1/m} k^s L_m^{-s} L_n^s)). \quad (4.29)$$

Since $kL_m^{-1}L_n \leq N_m L_m^{-1}L_n = L_{m-1}^{-1}L_n \leq 1$, we have $kL_m^{-1}L_n < e^{1/m} k^s L_m^{-s} L_n^s$. Therefore (4.29) and thus the left hand side of (4.28) tend to zero as $t \rightarrow \infty$. From this the Corollary follows. \square

It remains to prove the three lemmas. We will always assume that $t \geq 0$.

Proof of Lemma 4.4.9. The combination of Corollary 4.4.5 and (the second part of)

Lemma 4.4.7 for some $u \in V_{m-1}$ gives that

$$\mathbb{E} e^{tb(S_n)} \leq 1 - L_{m-1}^{s-1} + L_{m-1}^{s-1} (1 - N_m^{s-1} + N_m^{s-1} \exp(tL_m^{-s} L_n^s e^{2tL_{m+1}^{-s} L_n^s}))^k$$

if $t \leq L_{m+1}^s L_n^{-s}$, since $b(V_n)/c_u(V_n) = kL_m^{-1} L_n / (L_{m-1}^{-1} L_n) = kN_m^{-1}$. Therefore clearly

$$\mathbb{E} e^{tb(S_n)} \leq (1 - N_m^{s-1} + N_m^{s-1} \exp(tL_m^{-s} L_n^s e^{2tL_{m+1}^{-s} L_n^s}))^k \quad (4.30)$$

if $t \leq L_{m+1}^s L_n^{-s}$. Set

$$t = e^{-1/(2m)} (\log N_m^{1-s}) L_m^s L_n^{-s}.$$

We may clearly assume that the sequence (N_i) is increasing sufficiently rapidly to ensure $L_{m+1}^s L_n^{-s} \geq t \geq L_m^s L_n^{-s} \geq k^{-s} L_m^s L_n^{-s}$ for this particular t . Now

$$2tL_{m+1}^{-s} L_n^s = 2e^{-1/(2m)} (\log N_m^{1-s}) N_{m+1}^{-s} \leq 2(\log N_m^{1-s}) N_{m+1}^{-s};$$

we may assume that the right hand side is at most $1/(2m)$ and thus $e^{2tL_{m+1}^{-s} L_n^s} \leq e^{1/(2m)}$. Using this upper bound and (4.30), the left hand side of (4.25) can be estimated as

$$\begin{aligned} &\leq \frac{\left(1 - N_m^{s-1} + N_m^{s-1} \exp\left(e^{-1/(2m)} (\log N_m^{1-s}) e^{1/(2m)}\right)\right)^k}{\exp\left(e^{-1/(2m)} (\log N_m^{1-s}) e^{1/m} k^s\right)} e^{1/m} k^{s-1} L_m^{1-s} \\ &\leq \frac{(1 - N_m^{s-1} + N_m^{s-1} N_m^{1-s})^k}{\exp\left(e^{1/(2m)} (\log N_m^{1-s}) k^s\right)} e^{1/m} k^{s-1} L_m^{1-s} \\ &\leq \frac{2^k}{N_m^{(1-s)e^{1/(2m)}}} e^{1/m} k^{s-1} L_m^{1-s}. \end{aligned} \quad (4.31)$$

We may suppose that $\sqrt{\log N_m} \leq \frac{1-s}{4m} \log N_m$, thus $2^k \leq e^k \leq N_m^{(1-s)/(4m)}$. We may also suppose that $L_{m-1} \leq N_m^{1/(8m)}$, thus $L_m^{1-s} \leq N_m^{(1-s)(1+\frac{1}{8m})}$. Using these estimates, we may continue (4.31) as

$$\leq N_m^{-(1-s)(1+\frac{1}{4m})} e^{1/m} k^{s-1} N_m^{(1-s)(1+\frac{1}{8m})} \leq 3N_m^{-(1-s)/(8m)}. \quad \square$$

Proof of Lemma 4.4.10. We may use the inequality (4.30); that is,

$$\mathbb{E} e^{tb(S_n)} \leq (1 - N_m^{s-1} + N_m^{s-1} \exp(tL_m^{-s} L_n^s e^{2tL_{m+1}^{-s} L_n^s}))^k \quad (4.32)$$

when $t \leq L_{m+1}^s L_n^{-s}$. Set

$$t = \frac{\log N_m}{\log \log N_m} L_m^s L_n^{-s}.$$

Clearly this t has both the lower bound $t \geq k^{-s} L_m^s L_n^{-s}$ and the upper bound $t \leq L_{m+1}^s L_n^{-s}$ if we impose some mild conditions on the sequence (N_i) . Therefore, by using (4.32), the left hand side of (4.26) can be estimated as

$$\begin{aligned} &\leq \frac{\left(1 - N_m^{s-1} + N_m^{s-1} \exp((\log N_m)/(\log \log N_m) e^2)\right)^k}{\exp((\log N_m)/(\log \log N_m) e^{1/m} k^s)} e^{1/m} k^{s-1} L_m^{1-s} \\ &\leq \frac{\left(1 - N_m^{s-1} + N_m^{s-1} N_m^{e^2/\log \log N_m}\right)^k}{N_m^{e^{1/m} k^s / \log \log N_m}} e^{1/m} k^{s-1} L_m^{1-s} \\ &\leq \frac{\exp(k N_m^{s-1} N_m^{e^2/\log \log N_m})}{N_m^{e^{1/m} k^s / \log \log N_m}} e^{1/m} k^{s-1} L_m^{1-s} \\ &\leq \frac{\exp(N_m^{(s-1)/2} N_m^{e^2/\log \log N_m})}{N_m^{(\log N_m)^{s/2} / \log \log N_m}} e^{1/m} k^{s-1} L_m^{1-s}. \end{aligned} \quad (4.33)$$

We may clearly assume that $\frac{s-1}{2} + \frac{e^2}{\log \log N_m} < 0$ and that $\log \log N_m \leq (\log N_m)^{s/4}$, hence we may continue (4.33) as

$$\leq \frac{\exp(1)}{N_m^{(\log N_m)^{s/4}}} e^{1/m} k^{s-1} L_m^{1-s} \leq e^2 N_m^{-(\log N_m)^{s/4}} k^{s-1} L_m^{1-s} \leq 8 N_m^{-(\log N_m)^{s/4}} L_m^{1-s}.$$

Assuming that (N_i) is increasing sufficiently rapidly we have $L_m^{1-s} \leq N_m$, which concludes the proof. \square

Proof of Lemma 4.4.11. The combination of Corollary 4.4.5 and (the first part of) Lemma 4.4.7 for some $u \in V_{m-1}$ gives that

$$\mathbb{E} e^{tb(S_n)} \leq 1 - L_{m-1}^{s-1} + L_{m-1}^{s-1} \left(\exp(t L_{m-1}^{-s} L_n^s e^{2t L_m^{-s} L_n^s})\right)^{k N_m^{-1}}$$

if $t \leq L_m^s L_n^{-s}$, since $b(V_n)/c_u(V_n) = k L_m^{-1} L_n / (L_{m-1}^{-1} L_n) = k N_m^{-1}$. Therefore clearly

$$\mathbb{E} e^{tb(S_n)} \leq \exp(k N_m^{-1} t L_{m-1}^{-s} L_n^s e^{2t L_m^{-s} L_n^s}) \quad (4.34)$$

if $t \leq L_m^s L_n^{-s}$. Set

$$t = \frac{1}{4m} L_m^s L_n^{-s}.$$

We may clearly assume that $4m \leq N_m^{s(1-s)/2}$, therefore t has the lower bound

$k^{-s}L_m^sL_n^{-s}$. Using (4.34) we may estimate the left hand side of (4.27) as

$$\begin{aligned} &\leq \frac{\exp(kN_m^{s-1}e^{2/(4m)}/(4m))}{\exp(e^{1/m}k^s/(4m))} e^{1/m}k^{s-1}L_m^{1-s} \\ &\leq \exp\left(kN_m^{s-1}\frac{e^{1/(2m)}}{4m} - k^s\frac{e^{1/m}}{4m}\right) e^{1/m}k^{s-1}L_m^{1-s}. \end{aligned} \quad (4.35)$$

It is easy to check that $e^{1/m} - e^{1/(2m)} \geq 1/(2m)$. Since $k \leq N_m$, we have $kN_m^{s-1} \leq k^s$, and therefore

$$kN_m^{s-1}\frac{e^{1/(2m)}}{4m} - k^s\frac{e^{1/m}}{4m} \leq -\frac{k^s}{8m^2}.$$

Using this inequality and that $k \geq N_m^{(1-s)/2}$, we can continue (4.35) as

$$\leq \exp\left(-\frac{k^s}{8m^2}\right) e^{1/m}k^{s-1}L_m^{1-s} \leq 3 \exp\left(-\frac{N_m^{s(1-s)/2}}{8m^2}\right) L_m^{1-s}.$$

Assuming that (N_i) is increasing sufficiently rapidly we have $L_m^{1-s} \leq N_m$, which concludes the proof. \square

4.5 Main results

The first lemma is the only statement in this section which uses the concepts and results of the other sections of this chapter; that is, the random subtree of §4.1, the upper estimate of §4.2 and the lower estimate of §4.3.

4.5.1 Results on the real line

Lemma 4.5.1. *Let $K \subset \mathbb{R}$ be a compact set of Lebesgue measure 1, and let $f : K \setminus \{\max K\} \rightarrow [0, 1]$ be a continuous function. Then for every $0 < s < 1$, there exists a compact set $C \subset [0, 1]$ such that*

$$\mathcal{H}^s(C) \leq \mathcal{H}^s(f^{-1}(C)) \quad (4.36)$$

where $\mathcal{H}^s(C) < \infty$ and $\mathcal{H}^s(f^{-1}(C)) > 0$.

Remark 4.5.2. By requesting that $\mathcal{H}^s(C) < \infty$ and $\mathcal{H}^s(f^{-1}(C)) > 0$, we exclude the two possibilities $0 \leq 0$ and $\infty \leq \infty$ in inequality (4.36). The Lemma would be trivial if we did not exclude these possibilities. Note that we do not exclude the case when $0 = \mathcal{H}^s(C) \leq \mathcal{H}^s(f^{-1}(C)) = \infty$.

Proof of Lemma 4.5.1. We may suppose that the preimage of each point has Lebesgue

measure zero, otherwise the statement is trivial (namely, one can choose C to be a single point the preimage of which is of positive Lebesgue measure).

We define a system of intervals in $[0, 1]$ in the following way. Define $\varphi : [0, 1] \rightarrow \mathbb{R}$ as

$$\varphi(y) = \lambda(f^{-1}([0, y])).$$

Hence φ is a continuous and increasing function, and its range is $[0, 1]$.

Recall the tree V and the random subtree S of §4.1. For each non-negative integer n and $v = (i_1, \dots, i_n) \in V_n$ define U_v to be the closed interval with left endpoint

$$\max \left\{ y \in [0, 1] : \varphi(y) = \sum_{j=1}^n \frac{i_j}{L_j} \right\}$$

and right endpoint

$$\min \left\{ y \in [0, 1] : \varphi(y) = \frac{1}{L_n} + \sum_{j=1}^n \frac{i_j}{L_j} \right\}.$$

Thus, for example, $U_\emptyset = [\max \varphi^{-1}(0), \min \varphi^{-1}(1)]$. Therefore for two vertices u and v we have $U_u \subset U_v$ if $v \leq u$; and U_u and U_v have at most one point in common if u and v are incomparable. We also have

$$\lambda(f^{-1}(U_v)) = 1/L_n \tag{4.37}$$

for every $v \in V_n$.

Define the random compact sets C_n as

$$C_n = \bigcup \{U_v : v \in S_n\}$$

and let

$$C = \bigcap_{n=0}^{\infty} C_n.$$

The intervals U_v satisfy the assumptions of the upper estimate (§4.2). Hence from Proposition 4.2.1 we obtain

$$\mathbb{E} \mathcal{H}^s(C) \leq 1. \tag{4.38}$$

Set $\psi : K \rightarrow [0, 1]$ by

$$\psi(x) = \lambda((-\infty, x] \cap K).$$

Then ψ is a 1-Lipschitz function from K onto $[0, 1]$ preserving the Lebesgue measure.

For each $v \in V$, set

$$M_v = \psi(f^{-1}(U_v) \cup \{\max K\}).$$

Then (4.37) implies that $\lambda(M_v) = 1/L_n$ for every $v \in V_n$. It is easy to check that $\lambda(M_u \cap M_v) = 0$ if u and v are incomparable, and that $M_u \subset M_v$ if $v \leq u$. It is also easy to see that the sets $f^{-1}(U_v) \cup \{\max K\}$ are compact, thus the sets M_v are compact as well. Therefore the assumptions of the lower estimate (§4.3) are satisfied.

Set $D_n = \bigcup\{M_v : v \in S_n\}$, and set $D = \bigcap_n D_n$. We may apply Theorem 4.3.4 and obtain

$$\mathbb{E} \mathcal{H}^s(D) \geq 1. \quad (4.39)$$

Notice that $D_n = \psi(f^{-1}(C_n) \cup \{\max K\})$. Since ψ is monotonic, there can be only countably many points $y \in [0, 1]$ for which $\psi^{-1}(y)$ is not a single point. This implies that D and $\psi(\bigcap_n f^{-1}(C_n) \cup \{\max K\}) = \psi(f^{-1}(C) \cup \{\max K\})$ can differ only in countably many points. Since 1-Lipschitz functions do not increase Hausdorff measures, from (4.39) we deduce

$$\mathbb{E} \mathcal{H}^s(f^{-1}(C) \cup \{\max K\}) \geq 1.$$

Then we also have

$$\mathbb{E} \mathcal{H}^s(f^{-1}(C)) \geq 1. \quad (4.40)$$

From (4.38) we know that $\mathcal{H}^s(C)$ is almost surely finite, and (4.38) and (4.40) imply that

$$\mathbb{E}(\mathcal{H}^s(f^{-1}(C)) - \mathcal{H}^s(C)) \geq 0.$$

Combining this with (4.40) gives that with positive probability all the inequalities $\mathcal{H}^s(C) \leq \mathcal{H}^s(f^{-1}(C))$, $\mathcal{H}^s(C) < \infty$, $\mathcal{H}^s(f^{-1}(C)) > 0$ hold. \square

Proposition 4.5.3. *Let $A \subset \mathbb{R}$ be a Borel set of positive and finite Lebesgue measure, and let $f : A \rightarrow [0, 1]$ be a Borel mapping. Then for every $0 < s < 1$ there exists a compact set $C \subset [0, 1]$ such that $\mathcal{H}^s(C) < \infty$, $\mathcal{H}^s(f^{-1}(C)) > 0$ and*

$$\lambda(A)^s \mathcal{H}^s(C) \leq \mathcal{H}^s(f^{-1}(C)).$$

Proof. By applying a similarity transformation in the domain space, we may assume without loss of generality that $\lambda(A) = 1$.

By a repeated use of Luzin's theorem we can find *disjoint* compact sets $K_i \subset A$ ($i = 0, 1, \dots$) such that f is continuous on each K_i and that $\sum_{i=0}^{\infty} \lambda(K_i) = 1$. By the Lebesgue density theorem we may even suppose that the diameter of each K_i is at most twice its Lebesgue measure. Then we can find translation vectors $t_i \in \mathbb{R}$ such

that $K_i + t_i \subset [0, 3]$, and that $\max(K_i + t_i) < \min(K_{i+1} + t_{i+1})$. Put

$$K' = \bigcup_{i=0}^{\infty} (K_i + t_i).$$

Note that the closure of K' is $K' \cup \{\sup K'\}$. Let $\psi : \bigcup_i K_i \rightarrow K'$ be the unique bijection for which $\psi|_{K_i}$ is just the translation by t_i for each $i \in \mathbb{N}$. Define $g : K' \rightarrow [0, 1]$ by $g = f \circ \psi^{-1}$. It is easy to see that g is continuous. Thus we may apply Lemma 4.5.1 to $K = K' \cup \{\sup K'\}$ and the function g . We obtain a compact set $C \subset [0, 1]$ such that $\mathcal{H}^s(C) \leq \mathcal{H}^s(g^{-1}(C))$ where $\mathcal{H}^s(C) < \infty$ and $\mathcal{H}^s(g^{-1}(C)) > 0$. Since Hausdorff measures are translation invariant and σ -additive, the bijection ψ preserves Hausdorff measures. Hence $\mathcal{H}^s(g^{-1}(C)) = \mathcal{H}^s(f^{-1}(C))$ and the proof is finished. \square

The following auxiliary lemma will be used in the proof of Theorem 4.5.5. Similar statements surely exist in the literature, however, we could not find any reference.

Lemma 4.5.4. *Let μ and ν be atomless Borel measures on the unit circle \mathbb{T} such that $1 \leq \mu(\mathbb{T}) < \infty$ and $\mu(\mathbb{T}) \leq \nu(\mathbb{T})$. Then there exists a closed interval (an arc) $I \subset \mathbb{T}$ such that $\mu(I) = 1$ and $\nu(I) \geq 1$.*

Proof. If $\mu(\mathbb{T}) = 1$ then I can be chosen as \mathbb{T} . Therefore we may assume that $\mu(\mathbb{T}) > 1$.

Suppose first that $\nu(\mathbb{T}) < \infty$. Fix some $\delta > 0$. We define a Borel measure μ_δ on \mathbb{T} by setting $\mu_\delta = \mu + \delta\lambda$, where λ is the (normalized) Lebesgue measure on \mathbb{T} . Then $\mu_\delta(\mathbb{T}) = \mu(\mathbb{T}) + \delta$ and μ_δ possesses the property that every non-empty open set has positive μ_δ measure. For $x \in \mathbb{T}$ define $f(x)$ as the unique point in \mathbb{T} for which $\mu_\delta([x, f(x)]) = 1$ holds. Thus f is a $\mathbb{T} \rightarrow \mathbb{T}$ homeomorphism. We claim that there exists an interval $I_\delta \subset \mathbb{T}$ for which $\nu(I_\delta) \geq 1/(1 + \delta)$ and $\mu_\delta(I_\delta) = 1$. Indeed, put

$$E = \{(x, y) \in \mathbb{T}^2 : y \in [x, f(x)]\} = \{(x, y) \in \mathbb{T}^2 : x \in [f^{-1}(y), y]\}.$$

Then

$$\int_{\mathbb{T}} \nu([x, f(x)]) d\mu_\delta(x) = \mu_\delta \times \nu(E) = \int_{\mathbb{T}} \mu_\delta([f^{-1}(y), y]) d\nu(y) = \int_{\mathbb{T}} 1 d\nu(y) = \nu(\mathbb{T}).$$

Therefore there exists some $x \in \mathbb{T}$ such that $\nu([x, f(x)]) \mu_\delta(\mathbb{T}) \geq \nu(\mathbb{T})$. Then $\nu([x, f(x)]) \geq \nu(\mathbb{T})/(\mu(\mathbb{T}) + \delta) \geq 1/(1 + \delta)$, and we can let $I_\delta = [x, f(x)]$.

We can do the same and define I_δ for all $\delta > 0$. We can choose a sequence $\delta_n \rightarrow 0$ such that I_δ converges to some closed interval I (meaning that the left endpoints

converge to the left endpoint of I , and the right endpoints converge to the right endpoint of I). It is easy to check that $\mu(I) = \lim_n \mu(I_{\delta_n}) = \lim_n \mu_{\delta_n}(I_{\delta_n}) = 1$ and also that $\nu(I) = \lim_n \nu(I_{\delta_n}) \geq \lim_n 1/(1 + \delta_n) = 1$. Hence we are done with the case $\nu(\mathbb{T}) < \infty$.

Now suppose that $\nu(\mathbb{T}) = \infty$. Since $\mu(\mathbb{T}) < \infty$, it is easy to cover \mathbb{T} by finitely many closed intervals I_i such that $\mu(I_i) = 1$ for every i . One of these intervals must have infinite ν measure, we can choose that as I . \square

Theorem 4.5.5. *Let $f : [0, 1] \rightarrow [0, 1]$ be Borel. Then for every $0 < s < 1$ there exists a compact set C such that $\mathcal{H}^s(C) \leq 1$ and $\mathcal{H}^s(f^{-1}(C)) \geq 1$.*

Proof. This proof is based on a repeated use of Proposition 4.5.3. First we prove that there exists a compact set $C' \subset [0, 1]$ such that

$$\mathcal{H}^s(C') < \infty, \quad \mathcal{H}^s(f^{-1}(C')) \geq 1, \quad \text{and} \quad \mathcal{H}^s(C') \leq \mathcal{H}^s(f^{-1}(C')). \quad (4.41)$$

Suppose that such set C' does not exist.

We will define compact sets $C_k \subset [0, 1]$ for every positive integer k by a greedy algorithm such that each $f^{-1}(C_k)$ is of Lebesgue measure zero. Suppose that the sets C_i are already defined for $i = 1, \dots, k-1$. (Note that this automatically holds for $k = 1$.) Put

$$A_k = [0, 1] \setminus \bigcup_{i=1}^{k-1} f^{-1}(C_i);$$

this is a Borel set of Lebesgue measure 1. Apply Proposition 4.5.3 to the function $f|_{A_k}$. We conclude that there exists a compact set $C_k \subset [0, 1]$ such that

$$\mathcal{H}^s(C_k) < \infty, \quad \mathcal{H}^s(A_k \cap f^{-1}(C_k)) \geq 1/n_k \quad \text{and} \quad \mathcal{H}^s(C_k) \leq \mathcal{H}^s(A_k \cap f^{-1}(C_k)) \quad (4.42)$$

for some integer $n_k \geq 2$. Choose such a set C_k for which we may choose n_k to be minimal. Our hypothesis that (4.41) cannot be satisfied implies that we must have

$$\mathcal{H}^s(f^{-1}(C_k)) < 1. \quad (4.43)$$

Therefore $f^{-1}(C_k)$ is of Lebesgue measure zero.

It is easy to check that the sequence n_k is monotone increasing. We claim that n_k stabilizes (that is, it does not tend to infinity). Indeed, let $A_\infty = [0, 1] \setminus \bigcup_{i=1}^{\infty} f^{-1}(C_i)$. Apply Proposition 4.5.3 to the function $f|_{A_\infty}$ to obtain a compact set $C_\infty \subset [0, 1]$ such that

$$\mathcal{H}^s(C_\infty) < \infty, \quad \mathcal{H}^s(A_\infty \cap f^{-1}(C_\infty)) \geq 1/m \quad \text{and} \quad \mathcal{H}^s(C_\infty) \leq \mathcal{H}^s(A_\infty \cap f^{-1}(C_\infty))$$

for some positive integer m . It is easy to check $n_k \leq m$ holds for all k .

The sets $A_k \cap f^{-1}(C_k)$ are pairwise disjoint. Let $C' = C_1 \cup \dots \cup C_m$. The previous arguments imply that

$$\mathcal{H}^s(f^{-1}(C')) \geq \mathcal{H}^s\left(\bigcup_{i=1}^m (A_i \cap f^{-1}(C_i))\right) = \sum_{i=1}^m \mathcal{H}^s(A_i \cap f^{-1}(C_i)) \quad (4.44)$$

$$\geq \sum_{i=1}^m \frac{1}{n_i} \geq 1. \quad (4.45)$$

From (4.42) and (4.43) we obtain

$$\mathcal{H}^s(C') \leq \sum_{i=1}^m \mathcal{H}^s(C_i) \leq \sum_{i=1}^m \mathcal{H}^s(A_i \cap f^{-1}(C_i)) < m. \quad (4.46)$$

Combining (4.44) and (4.46) gives $\mathcal{H}^s(C') \leq \mathcal{H}^s(f^{-1}(C'))$. This, together with (4.45) and (4.46) imply that our hypothesis that (4.41) cannot be satisfied was false, thus there indeed exists a compact set $C' \subset [0, 1]$ such that (4.41) holds.

Now let $C' \subset [0, 1]$ be a compact set such that (4.41) holds. If $\mathcal{H}^s(C') \leq 1$ then we are done. Otherwise we define two Borel measures μ and ν on $[0, 2\pi)$ by setting

$$\mu(A) = \mathcal{H}^s(C' \cap A) \quad \text{and} \quad \nu(A) = \mathcal{H}^s(f^{-1}(C' \cap A))$$

for every Borel set $A \subset [0, 2\pi)$. The inequalities (4.41) imply that we may apply Lemma 4.5.4 after identifying $[0, 2\pi)$ with \mathbb{T} . We obtain *two* closed intervals $I_1 \subset [0, 1]$ and $I_2 \subset [0, 1]$ (one of them may be empty) such that for $C = C' \cap (I_1 \cup I_2)$ we have $\mathcal{H}^s(C) = 1$ and $\mathcal{H}^s(f^{-1}(C)) \geq 1$. This concludes the proof. \square

From Theorem 4.5.5 we can easily deduce the following.

Theorem 4.5.6. *Let $f : [0, 1] \rightarrow [0, 1]$ be Borel measurable. Then for every $0 < s < 1$ there exists a compact set $C \subset [0, 1]$ such that $\mathcal{H}^s(C) = 1$ and $\mathcal{H}^s(f(C)) \leq 1$.*

Proof. Let $f : [0, 1] \rightarrow [0, 1]$ be Borel measurable, and let $0 < s < 1$. Applying Theorem 4.5.5 we obtain a compact set $C \subset [0, 1]$ such that $\mathcal{H}^s(C) \leq 1$ and $\mathcal{H}^s(f^{-1}(C)) \geq 1$. Then we can choose a compact set $C' \subset f^{-1}(C)$ such that $\mathcal{H}^s(C') = 1$. Clearly $\mathcal{H}^s(f(C')) \leq 1$, thus the proof is finished. \square

Remark 4.5.7. Many steps of the previous proofs could be simplified if one wanted to prove only the following weaker version of Theorem 4.5.6. *Let $f : [0, 1] \rightarrow [0, 1]$ be Borel measurable, let $0 < s < 1$ and let $\varepsilon > 0$. Then there exists a compact set $C \subset [0, 1]$ such that $\mathcal{H}^s(C) = 1$ and $\mathcal{H}^s(f(C)) \leq 1 + \varepsilon$.* However, this weaker version

would leave open the question whether there exists a Borel map $f : [0, 1] \rightarrow [0, 1]$ such that $\mathcal{H}^s(f(B)) > \mathcal{H}^s(B)$ for every Borel set $B \subset [0, 1]$ satisfying $0 < \mathcal{H}^s(B) < \infty$.

Corollary 4.5.8. *Let $0 < s < 1$ and $c > 0$, $c \neq 1$ be fixed. Then the measures \mathcal{H}^s and $c \cdot \mathcal{H}^s$ are not Borel isomorphic on $[0, 1]$. (That is, there does not exist a Borel bijection $f : [0, 1] \rightarrow [0, 1]$ such that $\mathcal{H}^s(B) = c \mathcal{H}^s(f(B))$ for every Borel set $B \subset [0, 1]$.)* \square

Remark 4.5.9. Obviously, \mathcal{H}^s is Borel isomorphic to $c \mathcal{H}^s$ on the real line, as there is a similarity which realizes the isomorphism.

Remark 4.5.10. Corollary 4.5.8 suggests an interesting ‘compactness’ property of Borel measures. Let X be a Polish space, \mathcal{B} the Borel σ -algebra, and μ an atomless measure on \mathcal{B} . Let us say that μ is *compressible* if there exists a Borel map $f : X \rightarrow X$ such that

$$\mu(f^{-1}(B)) \leq \frac{\mu(B)}{2}$$

for every $B \in \mathcal{B}$. (There are many reasonable alternatives to this definition to describe roughly the same phenomenon.) Then Theorem 4.5.5 implies that \mathcal{H}^s restricted to $[0, 1]$ (to the Borel subsets of $[0, 1]$) is not compressible. However, \mathcal{H}^s on the real line is compressible (see Remark 4.5.9).

For σ -finite measures there is a very simple characterization of compressibility. If μ is σ -finite, then it is *not* compressible if and only if μ is finite and non-zero; that is, $0 < \mu(X) < \infty$. The non-trivial part of this statement follows from the well-known isomorphism theorem of σ -finite Borel measures; see for example [7, Theorem 17.41]. (That is, every atomless infinite σ -finite Borel measure on a Polish space is isomorphic to the Lebesgue measure on the real line, thus compressible.) Hence compressibility can be interesting only for non- σ -finite measures.

4.5.2 Results in Euclidean spaces

Proposition 4.5.11. *Let $A, B \subset \mathbb{R}^n$ be Borel sets satisfying $0 < \lambda(A) = \lambda(B) < \infty$, and let $f : A \rightarrow B$ be a Borel mapping. Let $0 < s < n$. Then for every $\varepsilon > 0$ there exists a Borel set $C \subset B$ such that $\mathcal{H}^s(C) \leq (1 + \varepsilon)\mathcal{H}^s(f^{-1}(C))$, where $\mathcal{H}^s(C) < \infty$ and $\mathcal{H}^s(f^{-1}(C)) > 0$.*

Lemma 4.5.12. *Let $F \subset \mathbb{R}^n$ be a compact set of positive Lebesgue measure, and let $\delta > 0$ be given. Let $Q \subset \mathbb{R}^n$ be an open cube of volume $(1 + \delta)\lambda(F)$. Then there exists an injective Borel mapping $\psi : F \rightarrow Q$ which preserves Lebesgue measure and all Hausdorff measures.*

Proof. As Lebesgue and Hausdorff measures are isometry invariant, we may assume without loss of generality that Q is an axis-parallel cube of volume $(1 + \delta)\lambda(F)$. We will show that F can be divided into finitely many Borel subsets so that we can choose ψ to be a translation on each of these subsets. Since Lebesgue and Hausdorff measures are translation invariant and additive, this will give the proof.

Let $r > 0$ be some small number, and consider a covering of \mathbb{R}^n by grid cubes $[0, r)^n + rz$, where $z \in \mathbb{Z}^n$. For sufficiently small r , the compact set F intersects at most $\lambda(F)r^{-n}(1 + \delta/2)$ grid cubes. Therefore these grid cubes can be translated to fit in the cube Q (if r is again sufficiently small compared to δ). From this we conclude that the sought injective Borel mapping $\psi : F \rightarrow Q$ exists. \square

Remark 4.5.13. The proof of Proposition 4.5.11 is based on Proposition 4.5.3, using Lemma 3.7.3 and Lemma 4.5.12. If in the statement of Proposition 4.5.11 we allowed $1 + \varepsilon$ to be some large constant depending on n and s , then the proof would be rather straightforward (but still lengthy). However, to get $1 + \varepsilon$, first we have to find a way to iterate f in some sense before reducing the problem to the one dimensional case (that is, Proposition 4.5.3).

Proposition 4.5.11 should be true even for $\varepsilon = 0$, but the author does not know how to prove that, and we will not need that for the applications.

Proof of Proposition 4.5.11. Consider n and s fixed. Let $c_{n,s}$ be the constant given by Lemma 3.7.3. Let $k > 0$ be so large that

$$k^{s/n}c_{n,s}^2 \geq 2. \quad (4.47)$$

There are two possibilities.

(i) Either there exists a Borel set $A' \subset A$ such that

$$\lambda(A') > k\lambda(f(A'));$$

(ii) or for every Borel set $B' \subset \mathbb{R}^n$,

$$\lambda(f^{-1}(B')) \leq k\lambda(B'). \quad (4.48)$$

First we deal with case (i). We do not need the extra ε in the Proposition in this case. We can choose a compact set $A_* \subset A'$ such that f restricted to A_* is continuous and that

$$\lambda(A_*) > k\lambda(f(A_*)) \geq k\lambda(f(A')). \quad (4.49)$$

Let f_* be the restriction of f to A_* .

Let us fix some small $\delta > 0$. Let Q_A be an open cube of volume $(1 + \delta)\lambda(A_*)$, and Q_B be an open cube of volume $(1 + \delta)\lambda(A_*)/k$. Using Lemma 4.5.12 (and (4.49)), fix injective Borel functions $\psi_A : A_* \rightarrow Q_A$ and $\psi_B : f_*(A_*) \rightarrow Q_B$ preserving Lebesgue and Hausdorff measures. Let $s_A : Q_A \rightarrow (0, 1)^n$ and $s_B : Q_B \rightarrow (0, 1)^n$ be surjective similarity transformations. Also consider the Borel set $E_n \subset [0, 1)$ and Borel bijection $p_n : E_n \rightarrow [0, 1)^n$ given by Lemma 3.7.3. Let g be a Borel mapping from $p_n^{-1}(s_A(\psi_A(A_*))) \subset [0, 1)$ to $[0, 1)$ defined by

$$g = p_n^{-1} \circ s_B \circ \psi_B \circ f_* \circ \psi_A^{-1} \circ s_A^{-1} \circ p_n.$$

Applying Proposition 4.5.3 to the function g and dimension s/n gives a compact set $C' \subset [0, 1]$ such that

$$\lambda(p_n^{-1}(s_A(\psi_A(A_*))))^{s/n} \mathcal{H}^{s/n}(C') \leq \mathcal{H}^{s/n}(g^{-1}(C')), \quad (4.50)$$

where $\mathcal{H}^{s/n}(C') < \infty$ and $\mathcal{H}^{s/n}(g^{-1}(C')) > 0$. It is easy to check that

$$\lambda(p_n^{-1}(s_A(\psi_A(A_*)))) = (1 + \delta)^{-1} \lambda(A_*)^{-1} \lambda(A_*) = (1 + \delta)^{-1}. \quad (4.51)$$

We define a Borel set

$$C = \psi_B^{-1} \circ s_B^{-1} \circ p_n(C') \subset f_*(A_*).$$

Then

$$\begin{aligned} \mathcal{H}^s(C) &\leq \mathcal{H}^s(s_B^{-1}(p_n(C'))) \leq \lambda(Q_B)^{s/n} \mathcal{H}^s(p_n(C')) \\ &\leq \lambda(Q_B)^{s/n} c_{n,s}^{-1} \mathcal{H}^{s/n}(C') = (1 + \delta)^{s/n} (\lambda(A_*)/k)^{s/n} c_{n,s}^{-1} \mathcal{H}^{s/n}(C') \end{aligned} \quad (4.52)$$

where the third inequality was obtained by (3.10). On the other hand,

$$g^{-1}(C') = p_n^{-1}(s_A(\psi_A(f_*^{-1}(C')))),$$

and thus

$$\begin{aligned} \mathcal{H}^{s/n}(g^{-1}(C')) &\leq c_{n,s}^{-1} \mathcal{H}^s(s_A(\psi_A(f_*^{-1}(C)))) = c_{n,s}^{-1} \lambda(Q_A)^{-s/n} \mathcal{H}^s(\psi_A(f_*^{-1}(C))) \\ &= c_{n,s}^{-1} \lambda(Q_A)^{-s/n} \mathcal{H}^s(f_*^{-1}(C)) \\ &= c_{n,s}^{-1} (1 + \delta)^{-s/n} \lambda(A_*)^{-s/n} \mathcal{H}^s(f_*^{-1}(C)). \end{aligned} \quad (4.53)$$

Combining (4.50) with (4.51), (4.52) and (4.53) gives that

$$(1 + \delta)^{-s/n} k^{s/n} c_{n,s}^2 \mathcal{H}^s(C) \leq \mathcal{H}^s(f_*^{-1}(C)), \quad (4.54)$$

where $\mathcal{H}^s(C) < \infty$ and $\mathcal{H}^s(f_*^{-1}(C)) > 0$. We may assume that $\delta < 1$, thus from (4.47) we obtain $(1 + \delta)^{-s/n} k^{s/n} c_{n,s}^2 \geq 1$. Therefore (4.54) implies

$$\mathcal{H}^s(C) \leq \mathcal{H}^s(f_*^{-1}(C)) \leq \mathcal{H}^s(f^{-1}(C)),$$

since f_* was just a restriction of f . Hence we are done with case (i).

Now let us assume that the second case holds: for every Borel set $B' \subset \mathbb{R}^n$, $\lambda(f^{-1}(B')) \leq k\lambda(B')$.

Let N be some large positive integer depending on $\varepsilon > 0$ given in the Proposition. Let $\delta > 0$ be very small depending on N . (The right choice of these constants will become clear during the proof.)

Let A_* be a compact subset of A such that $\lambda(A_*) = (1 - \delta)\lambda(A)$, and B_* be a compact subset of B such that $\lambda(B_*) = (1 - \delta)\lambda(A)$. Let $Q \subset \mathbb{R}^n$ be an open cube of volume $\lambda(A)$. Using Lemma 4.5.12, fix injective Borel functions $\psi_A : A_* \rightarrow Q$ and $\psi_B : B_* \rightarrow Q$ which preserve Lebesgue and Hausdorff measures. We define a Borel map

$$g : \psi_A(A_* \cap f^{-1}(B_*)) \rightarrow Q$$

by setting

$$g = \psi_B \circ f \circ \psi_A^{-1}. \quad (4.55)$$

Let $D = \psi_A(A_* \cap f^{-1}(B_*))$ be the domain of g . Note that $D \subset Q$. We claim that this domain is in fact a large portion of the cube Q . Indeed,

$$\begin{aligned} \lambda(D) &= \lambda(A_* \cap f^{-1}(B_*)) \\ &= \lambda(A_* \setminus f^{-1}(B \setminus B_*)) \geq \lambda(A_*) - \lambda(f^{-1}(B \setminus B_*)) \\ &\geq \lambda(A_*) - k\lambda(B \setminus B_*) = (1 - \delta)\lambda(A) - k\delta\lambda(A) \\ &= (1 - (k + 1)\delta)\lambda(A) = (1 - (k + 1)\delta)\lambda(Q). \end{aligned} \quad (4.56)$$

Since ψ_A and ψ_B are Lebesgue measure preserving (where they are defined), it is easy to check that (4.48) implies that $\lambda(g^{-1}(B')) \leq k\lambda(B')$ for all Borel sets $B' \subset \mathbb{R}^n$.

For each positive integer j we have

$$\lambda(g^{-j}(Q \setminus D)) \leq k^j \lambda(Q \setminus D) \leq k^j (k + 1)\delta \lambda(Q)$$

by (4.56). Therefore

$$\lambda\left(\bigcup_{j=1}^{N-1} g^{-j}(Q \setminus D)\right) \leq k^N(k+1)\delta\lambda(Q) \quad (4.57)$$

since $k \geq 2$ by (4.47). Set

$$D' = D \cap g^{-1}(D) \cap g^{-2}(D) \cap \dots \cap g^{-(N-1)}(D).$$

The set D' is the largest domain on which g^N is defined. It is easy to check that

$$D' = D \setminus (g^{-1}(Q \setminus D) \cup g^{-2}(Q \setminus D) \cup \dots \cup g^{-(N-1)}(Q \setminus D)).$$

The estimate (4.57) implies that $\lambda(D') \geq \lambda(D) - k^N(k+1)\delta\lambda(Q)$. Combining this with (4.56), for sufficiently small δ we have

$$\lambda(D') \geq \lambda(Q)/2. \quad (4.58)$$

Now consider a surjective similarity map $s_Q : Q \rightarrow (0, 1)^n$. Let $E_n \subset [0, 1)$ and $p_n : E_n \rightarrow [0, 1)^n$ be the Borel set and the bijection of Lemma 3.7.3. We define a function

$$h : p_n^{-1}(s_Q(D')) \rightarrow [0, 1)$$

by setting

$$h = p_n^{-1} \circ s_Q \circ g^N \circ s_Q^{-1} \circ p_n.$$

Applying Proposition 4.5.3 to the function h and dimension s/n gives a compact set $C' \subset [0, 1]$ such that

$$\lambda(p_n^{-1}(s_Q(D')))^{s/n} \mathcal{H}^{s/n}(C') \leq \mathcal{H}^{s/n}(h^{-1}(C')), \quad (4.59)$$

where $\mathcal{H}^{s/n}(C') < \infty$ and $\mathcal{H}^{s/n}(h^{-1}(C')) > 0$. Clearly $\lambda(p_n^{-1}(s_Q(D'))) = \lambda(s_Q(D')) = \lambda(Q)^{-1}\lambda(D) \geq 1/2$ by (4.58).

Define the Borel set $C'' = s_Q^{-1}(p_n(C'))$. Then

$$\mathcal{H}^s(C'') \leq \lambda(Q)^{s/n} \mathcal{H}^s(p_n(C')) \leq \lambda(Q)^{s/n} c_{n,s}^{-1} \mathcal{H}^{s/n}(C'). \quad (4.60)$$

Since $h^{-1}(C') = p_n^{-1}(s_Q(g^{-N}(C'')))$, we have

$$\mathcal{H}^{s/n}(h^{-1}(C')) \leq c_{n,s}^{-1} \mathcal{H}^s(s_Q(g^{-N}(C''))) = c_{n,s}^{-1} \lambda(Q)^{-s/n} \mathcal{H}^s(g^{-N}(C'')). \quad (4.61)$$

Combining (4.59) with (4.60) and (4.61) gives that

$$\mathcal{H}^s(C'') \leq 2c_{n,s}^{-2} \mathcal{H}^s(g^{-N}(C'')),$$

where $\mathcal{H}^s(C'') < \infty$ and $\mathcal{H}^s(g^{-N}(C'')) > 0$. Then there must exist some $j \in \{0, 1, \dots, N-1\}$ such that

$$\mathcal{H}^s(g^{-j}(C'')) \leq (2c_{n,s}^{-2})^{1/N} \mathcal{H}^s(g^{-(j+1)}(C''))$$

and that $\mathcal{H}^s(g^{-j}(C'')) < \infty$ and $\mathcal{H}^s(g^{-(j+1)}(C'')) > 0$ (as easily verified). Set $C''' = g^{-j}(C'')$ for such a j . We may choose N large enough (at the beginning of the proof) to ensure $(2c_{n,s}^{-2})^{1/N} \leq (1 + \varepsilon)$. Thus

$$\mathcal{H}^s(C''') \leq (1 + \varepsilon) \mathcal{H}^s(g^{-1}(C''')) \quad (4.62)$$

and $\mathcal{H}^s(C''') < \infty$ and $\mathcal{H}^s(g^{-1}(C''')) > 0$. Recall that the domain of $g = \psi_B \circ f \circ \psi_A^{-1}$ is $D = \psi_A(A_* \cap f^{-1}(B_*))$, see (4.55). Therefore we have

$$g^{-1}(C''') = \psi_A(A_* \cap f^{-1}(\psi_B^{-1}(C'''))).$$

Setting $C = \psi_B^{-1}(C''')$ gives $g^{-1}(C''') = \psi_A(A_* \cap f^{-1}(C))$. Since ψ_A and ψ_B preserve Hausdorff measures, (4.62) implies that

$$\mathcal{H}^s(C) \leq \mathcal{H}^s(C''') \leq (1 + \varepsilon) \mathcal{H}^s(A_* \cap f^{-1}(C)) \leq (1 + \varepsilon) \mathcal{H}^s(f^{-1}(C)).$$

From the previous arguments it is also clear that $\mathcal{H}^s(C) < \infty$ and $\mathcal{H}^s(f^{-1}(C)) > 0$, therefore the proof is finished. \square

By applying suitable similarity transformations in the domain and range spaces, it is easy to deduce the following generalization of Proposition 4.5.11.

Proposition 4.5.14. *Let $A, B \subset \mathbb{R}^n$ be Borel sets of positive and finite Lebesgue measures, and let $f : A \rightarrow B$ be a Borel mapping. Let $0 < s < n$. Then for every $\varepsilon > 0$ there exists a Borel set $C \subset B$ such that*

$$\lambda(A)^s \mathcal{H}^s(C) \leq (1 + \varepsilon) \lambda(B)^s \mathcal{H}^s(f^{-1}(C)),$$

where $\mathcal{H}^s(C) < \infty$ and $\mathcal{H}^s(f^{-1}(C)) > 0$. \square

Theorem 4.5.15. *Let $D \subset \mathbb{R}^n$ be Borel and let $f : D \rightarrow \mathbb{R}^n$ be a Borel mapping. Let $0 < s < n$ be fixed. If f does not decrease the \mathcal{H}^s measure of any sets, then it*

does not decrease the Lebesgue measure either. That is, if we have

$$\mathcal{H}^s(A) \leq \mathcal{H}^s(f(A))$$

for every Borel set $A \subset D$, then

$$\lambda(A) \leq \lambda(f(A))$$

for every Borel set $A \subset D$.

Remark 4.5.16. It is easy to check that Theorem 4.5.15 can be equivalently formulated in the following way. Let $D \subset \mathbb{R}^n$ be Borel and let $f : D \rightarrow \mathbb{R}^n$ be a Borel mapping. Let $0 < s < n$ be fixed. If we have $\mathcal{H}^s(f^{-1}(B)) \leq \mathcal{H}^s(B)$ for every Borel set $B \subset \mathbb{R}^n$, then we also have $\lambda(f^{-1}(B)) \leq \lambda(B)$ for every Borel set $B \subset \mathbb{R}^n$.

Proof of Theorem 4.5.15. Suppose to the contrary that there exists a Borel set $A \subset D$ such that $\lambda(A) > \lambda(f(A))$. We may suppose that $\lambda(A) < \infty$, otherwise we may work with a subset of A instead. There exist an expanding similarity ψ of \mathbb{R}^n and a Borel set $B \supset \psi(f(A))$ such that $\lambda(B) = \lambda(A)$. Let $\varepsilon > 0$ be so small that $(1 + \varepsilon)^{1/s}$ is smaller than the similarity ratio of ψ . Applying Proposition 4.5.11 to the function $\psi \circ f|_A : A \rightarrow B$ gives us a Borel set C such that

$$\begin{aligned} \mathcal{H}^s(C) &\leq (1 + \varepsilon)\mathcal{H}^s((f|_A)^{-1}(\psi^{-1}(C))) \\ &= (1 + \varepsilon)\mathcal{H}^s(A \cap f^{-1}(\psi^{-1}(C))) \\ &\leq (1 + \varepsilon)\mathcal{H}^s(f^{-1}(\psi^{-1}(C))), \end{aligned} \tag{4.63}$$

where $\mathcal{H}^s(C) < \infty$ and $\mathcal{H}^s(f^{-1}(\psi^{-1}(C))) > 0$.

Let $C' = \psi^{-1}(C)$. Then $\mathcal{H}^s(C) > (1 + \varepsilon)\mathcal{H}^s(C')$ or $\mathcal{H}^s(C') = 0$. In either case, from (4.63) we obtain

$$\mathcal{H}^s(C') < \mathcal{H}^s(f^{-1}(C')).$$

This contradicts the assumptions of the theorem for the Borel set $f^{-1}(C') \subset D$. \square

The author would like to thank David Preiss for some remarks which were essential for proving the following theorem.

Theorem 4.5.17. *Let $f : D \rightarrow \mathbb{R}^n$ be a Borel mapping on a Borel set $D \subset \mathbb{R}^n$, and let $0 < s < n$ be fixed. If f does not increase the \mathcal{H}^s measure of any sets, then it does not increase the Lebesgue measure either. That is, if*

$$\mathcal{H}^s(A) \geq \mathcal{H}^s(f(A)) \tag{4.64}$$

for every Borel set $A \subset D$, then

$$\lambda(A) \geq \lambda(f(A))$$

for every Borel set $A \subset D$.

Proof. Suppose to the contrary that there exists a Borel set $A \subset D$ for which $\lambda(A) < \lambda(f(A))$. Here $f(A)$ is analytic and therefore Lebesgue measurable. The Borel function $f|_A$ is not necessarily invertible, however, we would like to take at least some partial inverse of it. As David Preiss pointed out, we can always take a Lebesgue measurable function $g : f(A) \rightarrow A$ such that $f \circ g$ is the identity. This follows, for example, from the uniformization theorem of Jankov and von Neumann [7, Theorem 18.1].

Let us choose Borel sets $A' \supset A$ and $B \subset f(A)$ such that $0 < \lambda(A') < \lambda(B) < \infty$ and that $g|_B$ is Borel. (By Luzin's theorem, we could even require that B is compact and $g|_B$ is continuous.) Applying Proposition 4.5.14 to $g|_B : B \rightarrow A'$ gives us a Borel set $C \subset A'$ such that

$$\lambda(B)^s \mathcal{H}^s(C) \leq (1 + \varepsilon) \lambda(A')^s \mathcal{H}^s((g|_B)^{-1}(C)), \quad (4.65)$$

where $\mathcal{H}^s(C) < \infty$ and $\mathcal{H}^s((g|_B)^{-1}(C)) > 0$. Since $f \circ g$ is the identity, $(g|_B)^{-1}(C) \subset f(C)$. We may choose ε so small that $(1 + \varepsilon)\lambda(A')^s < \lambda(B)^s$, and then (4.65) implies that

$$\mathcal{H}^s(C) < \mathcal{H}^s(f(C)),$$

which contradicts (4.64). □

Remark 4.5.18. The analogue of Remark 4.5.16 does not apply to Theorem 4.5.17. In fact, if we have a Borel mapping $f : D \rightarrow \mathbb{R}^n$ on a Borel set $D \subset \mathbb{R}^n$, and $\mathcal{H}^s(f^{-1}(B)) \geq \mathcal{H}^s(B)$ for every Borel set $B \subset f(D)$, then we cannot conclude that $\lambda(f^{-1}(B)) \geq \lambda(B)$ for every Borel set $B \subset f(D)$. See Example 4.5.19 for a hint. See also Claim 4.5.20.

Example 4.5.19. Let $f : [0, 1) \rightarrow [0, 1)$ be the mapping $f(x) = \{2x\}$ (where $\{ \}$ denotes fractional part). Then f preserves the Lebesgue measure, but does not preserve \mathcal{H}^s measures; that is,

$$\lambda(f^{-1}(B)) = \lambda(B),$$

but

$$\mathcal{H}^s(f^{-1}(B)) = 2^{1-s} \mathcal{H}^s(B)$$

for every Borel set $B \subset [0, 1)$, for every $0 < s < 1$. □

Claim 4.5.20. *There exists a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f^{-1}(x)$ has Hausdorff dimension 1 for every $x \in \mathbb{R}$.*

Sketch of proof. It is easy to construct a compact set $A \subset \mathbb{R}$ of Hausdorff dimension 1 such that $A - A$ is nowhere dense. Then it is not hard to deduce that there exists a compact uncountable set $P \subset \mathbb{R}$ such that $A + p_1$ and $A + p_2$ are disjoint whenever $p_1, p_2 \in P$ are distinct. Fix a continuous surjective function $g : P \rightarrow [0, 1]$. Let $f_0 : A + P \rightarrow [0, 1]$ be the function defined by $f_0(a + p) = g(p)$ ($a \in A, p \in P$). This function is well-defined and continuous. Moreover, for every $x \in [0, 1]$, the set $f_0^{-1}(x)$ contains a translated copy of A thus it is of Hausdorff dimension 1. It is easy to extend f_0 to \mathbb{R} so that it possesses all the desired properties. \square

Theorem 4.5.21. *Let $n \geq 1$ and let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a Borel bijection. If f preserves the s -dimensional Hausdorff measure for some $0 < s < n$, then it also preserves the Lebesgue measure. That is, if*

$$\mathcal{H}^s(f^{-1}(B)) = \mathcal{H}^s(B)$$

for every Borel set $B \subset \mathbb{R}^n$, then

$$\lambda(f^{-1}(B)) = \lambda(B)$$

for every measurable set $B \subset \mathbb{R}^n$.

Proof. This is immediate from Theorem 4.5.15 and Theorem 4.5.17. Note that since f is injective, we do not need the uniformization theorem of Jankov and von Neumann this time. \square

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Summary

Geometric measure theory is concerned with investigating subsets of Euclidean spaces from measure theoretical point of view. Its basic tool is the Hausdorff measure. As it is well-known, for every $0 \leq s \leq n$ there exists a measure \mathcal{H}^s , the so-called s -dimensional Hausdorff measure on \mathbb{R}^n . Therefore, it is a fundamental question whether Hausdorff measures of different dimensions can be Borel isomorphic. This was an unsolved problem for several years, attributed to B. Weiss and popularized by D. Preiss.

Related to this problem, M. Elekes raised two questions which belong to the topic of restriction theorems.

In Chapter 2 we answer the questions of M. Elekes by showing that for every Lebesgue measurable function $f : [0, 1] \rightarrow \mathbb{R}$ there exists a compact set $C \subset [0, 1]$ of Hausdorff dimension $1/2$ such that $f|_C$ is of bounded variation; and we also prove that there exist compact sets $C_\alpha \subset [0, 1]$ of Hausdorff dimension $1 - \alpha$ such that $f|_{C_\alpha}$ is Hölder- α ($0 < \alpha < 1$). By the results of M. Elekes, these dimension bounds are sharp.

In Chapter 3 we solve the above mentioned problem of D. Preiss and B. Weiss by showing that Hausdorff measures of different dimensions are not Borel isomorphic. In fact, we show that for every Borel measurable function $f : \mathbb{R} \rightarrow \mathbb{R}$ and for every $0 < s < 1$ there exists a compact set C of Hausdorff dimension s such that $f(C)$ is of Hausdorff dimension at most s .

The proof of this theorem is based on two types of random constructions. One of them can be used to obtain a random Cantor set of dimension at most s almost surely; and the other to obtain a random compact set of dimension at least s almost surely.

In Chapter 4 we prove a quantitative sharp version of the previous result. Let $f : [0, 1] \rightarrow [0, 1]$ be Borel. Then for every $0 < s < 1$ there exists a compact set C such that $\mathcal{H}^s(C) = 1$ and $\mathcal{H}^s(f(C)) \leq 1$. This result implies that the measures \mathcal{H}^s and $c \cdot \mathcal{H}^s$ (where $c > 0$, $c \neq 1$) on the unit interval $[0, 1]$ are not Borel isomorphic.

Among others, we also prove the following consequences. Let $f : D \rightarrow \mathbb{R}^n$ be a Borel mapping on a Borel set $D \subset \mathbb{R}^n$, and let $0 < s < n$. If f does not increase the \mathcal{H}^s measure of any sets, then it does not increase the Lebesgue measure either.

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a Borel bijection. If f preserves the s -dimensional Hausdorff measure for some $0 < s < n$, then it also preserves the Lebesgue measure.

Magyar nyelvű összefoglalás

A geometriai mértékelmélet főként az euklideszi terek részhalmazait vizsgálja mértékelméleti szempontból. E vizsgálatok alapvető eszköze a Hausdorff-mérték. Ismert, hogy minden $0 \leq s \leq n$ számhoz tartozik egy \mathcal{H}^s mérték, az ún. s -dimenziós Hausdorff-mérték. Ezért alapvetően fontos annak az eldöntése, hogy ezek a mértékek ténylegesen különböznek-e, avagy lehetnek-e különböző dimenziós Hausdorff-mértékek Borel-izomorfak. Ezt a hosszú időn keresztül megoldatlan problémát B. Weiss vetette fel és D. Preiss népszerűsítette.

Ehhez a problémához kapcsolódik Elekes Márton két kérdése függvények megszorításairól.

A 2. fejezetben megválaszoljuk Elekes Márton kérdéseit megmutatva, hogy minden $f : [0, 1] \rightarrow \mathbb{R}$ Lebesgue-mérhető függvényhez létezik egy olyan $1/2$ Hausdorff-dimenziós $C \subset [0, 1]$ kompakt halmaz, hogy $f|_C$ korlátos változású; valamint minden $0 < \alpha < 1$ számhoz létezik egy olyan $1 - \alpha$ Hausdorff-dimenziós $C_\alpha \subset [0, 1]$ kompakt halmaz, hogy $f|_{C_\alpha}$ Hölder- α . Elekes Márton eredményei szerint ezek a tételek nem javíthatók.

A 3. fejezetben megválaszoljuk D. Preiss és B. Weiss kérdését: megmutatjuk, hogy különböző dimenziós Hausdorff-mértékek nem lehetnek Borel-izomorfak. Valójában azt bizonyítjuk, hogy minden $f : \mathbb{R} \rightarrow \mathbb{R}$ Borel-mérhető függvényhez és minden $0 < s < 1$ számhoz létezik egy C kompakt halmaz, amelynek Hausdorff-dimenziója s , és amelyre $f(C)$ Hausdorff-dimenziója legfeljebb s .

E tétel bizonyítása kétfajta véletlen konstrukción alapszik. Az egyik olyan véletlen Cantor-halmazokat ad, melyek dimenziója majdnem biztosan legfeljebb s . A másikkal pedig olyan véletlen kompakt halmazok kaphatók, amelyek dimenziója majdnem biztosan legalább s .

A 4. fejezetben az előző eredményt élesítjük. Legyen $f : [0, 1] \rightarrow [0, 1]$ Borel-mérhető. Ekkor minden $0 < s < 1$ számhoz létezik egy $C \subset [0, 1]$ kompakt halmaz, hogy $\mathcal{H}^s(C) = 1$ és $\mathcal{H}^s(f(C)) \leq 1$. Ebből a tételből következik, hogy a \mathcal{H}^s és $c \cdot \mathcal{H}^s$ mértékek (ahol $c > 0$ és $c \neq 1$) a $[0, 1]$ intervallumon nem Borel-izomorfak.

Az előzőből további tételeket is levezetünk. Legyen $f : D \rightarrow \mathbb{R}^n$ Borel-mérhető a $D \subset \mathbb{R}^n$ Borel-halmazon, és legyen $0 < s < n$. Ha f nem növeli egyetlen halmaz \mathcal{H}^s -mértékét sem, akkor nem növeli egyetlen halmaz Lebesgue-mértékét sem.

Végül megmutatjuk, hogy ha egy $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ Borel-bijekció megőrzi az s -dimenziós Hausdorff-mértéket valamely $0 < s < n$ -re (vagyis \mathcal{H}^s -mértéktartó), akkor a Lebesgue-mértéket is megőrzi.