Covering the real line with translates of a zero dimensional compact set

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Abstract

We construct a compact set C of Hausdorff dimension zero so that $cof(\mathcal{N})$ many translates of C cover the real line. Hence it is consistent with ZFC that less than continuum many translates of a zero dimensional compact set can cover the real line. This answers a question of Dan Mauldin.

1 Introduction

Given a set $A \subset \mathbb{R}$, how many of its translates are needed to cover the real line? It is clear that if A is of Lebesgue measure zero or if A is of the first Baire category, then countable many translates are not enough. Therefore if we assume the continuum hypothesis, then for these kind of sets clearly we need continuum many translates to cover the real line.

Gary Gruenhage showed that it is not possible to cover $\mathbb{R}$ with less than continuum many translates of the standard middle-third Cantor set. (This is a proof in ZFC, without any extra set theoretic assumptions.) He then raised the following question (see also [2] and [3]):

Is it consistent that there exist a compact set of Lebesgue measure zero and less than continuum many of its translates that cover $\mathbb{R}$?

Since this was unsolved for some time, Dan Mauldin asked whether at least the compact sets of Hausdorff dimension less than 1 possess the property that less than continuum many of their translates are not enough to cover $\mathbb{R}$. Udayan Darji and Tamás Keleti [2] showed in a beautiful simple proof that this is indeed true for compact sets of *packing* dimension less than 1.

Márton Elekes and Juris Steprāns solved the question of Gruenhage [3]. They showed that $\mathbb{R}$ can be covered by $cof(\mathcal{N})$ many translates of some compact set of measure zero of Erdős and Kakutani. Here $cof(\mathcal{N})$ is the so-called cofinality invariant of Lebesgue null sets, and it is consistent that $cof(\mathcal{N})$ is less than continuum (see e.g. [1]).

In this note we answer Dan Mauldin's question in the negative. We present a compact set C of Hausdorff dimension zero so that $\mathbb{R}$ can be covered by $cof(\mathcal{N})$ many translates of C (Theorem 2.2). Thus it is consistent with ZFC that $\mathbb{R}$ can be covered by less than continuum many translates of some compact set of Hausdorff dimension zero.

We remark here that if we do not restrict ourselves to compact sets then the previous statement is clear: If A is a residual subset of $\mathbb{R}$, and $B \subset \mathbb{R}$ is a set of the second Baire category, then clearly $A + B = \mathbb{R}$. Let us choose A to be a residual G_δ set of Hausdorff dimension zero (e.g. the set of Liouville numbers). It is consistent that there exists a set of the second Baire category consisting of less than continuum many points (see e.g. [1]), that set should be chosen as B. Therefore it is consistent that $\mathbb{R}$ can be covered by less than continuum many translates of a zero dimensional G_δ set.

In the next section we review how it is possible to write $\mathbb{R}$ as a union of $cof(\mathcal{N})$ many "small" compact sets and we state our main results. These are then proved in the third section.

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Notation and definitions. We denote the cofinality invariant of Lebesgue null sets by $cof(\mathcal{N})$, which is the smallest cardinal κ for which there exist κ many null sets so that every null set is contained in one of them.

We denote the d-dimensional Hausdorff pre-measure of a set $A \subset \mathbb{R}$ by $\mathcal{H}^d_\infty(A)$, defined as

$$\mathcal{H}^d_\infty(A) = \inf\{\sum_{i=1}^{\infty} (\operatorname{diam} I_i)^d : A \subset \cup_{i=1}^{\infty} I_i\}.$$

The Hausdorff dimension of A is then defined as $\inf\{d > 0 : \mathcal{H}^d_\infty(A) = 0\}$.

We denote the set of real numbers by $\mathbb{R}$ and the set of integers by $\mathbb{Z}$.

2 Covering $\mathbb{R}$ with small sets

Let ω denote the first infinite ordinal, that is, $\omega = \{0, 1, 2, \ldots\}$. Let $f : \omega \setminus \{0\} \to \omega \setminus \{0, 1\}$. A set of the form $S = \prod_{n=1}^{\infty} A_n$, where each A_n is of size at most $f(n)$, is called an *f-slalom* (by $\prod$ we mean the direct product of sets). It is known that if $\lim f(n) = \infty$, then ω^ω can be covered by $cof(\mathcal{N})$ many f-slaloms (Gruenhage and Levy [4, Theorem 2.12.] and [5], see also Bartoszyński and Judah [1]).

Hence it is clear that if we fix two functions $f, g : \omega \setminus \{0\} \to \omega \setminus \{0, 1\}$ where $\lim g(n) = \infty$, then every f-slalom can be covered by $cof(\mathcal{N})$ many g-slaloms. We describe below how it is possible to use this fact to cover the real line with $cof(\mathcal{N})$ many "small" compact sets (here we follow Elekes and Steprāns [3]). So let us take the f-slalom $T = \prod_{n=1}^{\infty} \{0, 1, \ldots, f(n)-1\}$, and let us cover it by $cof(\mathcal{N})$ many g-slaloms S_α ($\alpha < cof(\mathcal{N})$). We may suppose without loss of generality that each S_α is contained in T; that is, they are of the form $\prod_{n=1}^{\infty} A_n$ where $A_n \subset \{0, 1, \ldots, f(n) - 1\}$ and $|A_n| \le g(n)$ for each n. Then

$$[0, 1] = \{\sum_{n=1}^{\infty} \frac{s_n}{f(1)f(2)\ldots f(n)} : (s_n)_{n=1}^{\infty} \in T\}.$$

(It is worth thinking of s_n as digits of a numeral system with "increasing base" corresponding to the series $f(n)$.) For a slalom $S \subset T$ define

$$S^* = \{\sum_{n=1}^{\infty} \frac{s_n}{f(1)f(2)\ldots f(n)} : (s_n)_{n=1}^{\infty} \in S\}, \tag{1}$$

or equivalently, when $S = \prod_{n=1}^{\infty} A_n$,

$$S^* = (\prod_{n=1}^{\infty} A_n)^* = \frac{1}{f(1)} A_1 + \frac{1}{f(1)f(2)} A_2 + \frac{1}{f(1)f(2)f(3)} A_3 + \ldots. \tag{2}$$

(This infinite Minkowski sum is well-defined, and as S^* is a continuous image of the compact space $\prod A_n$, it is compact. For later reference note that $(\prod_{n=1}^{\infty} A_n)^*$ makes sense even if, say, $A_n \subset \{0, 1, \ldots, 2f(n) - 1\}$.)

Since the slaloms S_α cover T, the compact sets S^*_α ($\alpha < cof(\mathcal{N})$) cover the unit interval. These sets S^*_α are small if g is much smaller than f. That is, as it is easy to see, the Hausdorff dimension of the sets S^*_α is at most

$$\liminf_{n\to\infty} \frac{\log(g(1)\ldots g(n))}{\log(f(1)\ldots f(n))},$$

which can be zero if we choose g and f accordingly. Thus $[0, 1]$ (and also of course $\mathbb{R}$) can be covered by $cof(\mathcal{N})$ many compact sets of Hausdorff dimension zero. However, our point here is to find one compact set of Hausdorff dimension zero so that $\mathbb{R}$ can be covered by $cof(\mathcal{N})$ many translates of this compact set. What we will do is to find a compact set C of Hausdorff dimension zero so that for every g-slalom $S \subset T$, a translate of C will cover S^*. Then clearly $\mathbb{R}$ can be covered by $cof(\mathcal{N})$ many translates of C.

We will choose g to be (say) $g(n) = n + 1$, the function f will be determined implicitly by the construction.

Theorem 2.1. *Let $g(n) = n+1$. There exists a compact set C of Hausdorff dimension zero and a function $f : \omega \setminus \{0\} \to \omega \setminus \{0, 1\}$ so that for every g-slalom $S \subset \prod_{n=1}^{\infty}\{0, 1, \dots, f(n) - 1\}$, S^* can be covered by a translate of C.*

We prove this theorem in the next section. As it was shown above, this theorem immediately yields our main result.

Theorem 2.2. *There exists a compact set C of Hausdorff dimension zero so that $\mathbb{R}$ can be covered by $cof(\mathcal{N})$ many translates of C.*

3 Construction

Before we prove Theorem 2.1, we outline the construction of the set C in an informal way. So we need to construct a (small) set C so that every (very small) set $S^* \subset [0, 1]$ can be covered by a translate of C. First we will find compact sets $K_n \subset [0, 2]$ (each of which is a union of finitely many intervals) with arbitrarily small Hausdorff pre-measure (of arbitrarily small dimension) so that every $(n+1)$-point set $B \subset [0, 1]$ can be covered by a translate of K_n. Then to this set K_n we associate a large positive integer $f(n)$ and a set $A_n \subset \{0, 1, \dots, 2f(n) - 1\}$ with the property that every set $B_n \subset \{0, 1, \dots, f(n) - 1\}$ with $|B_n| \le n + 1$ can be covered by a translate of A_n. Setting $g(n) = n + 1$, this yields that for every g-slalom $\prod B_n$ the set $(\prod B_n)^*$ (as defined by (2)) can be covered by a translate of $(\prod A_n)^*$. Finally, we show that $(\prod A_n)^*$ is of dimension zero using the fact that the sets K_n can be chosen to be "arbitrarily small". Therefore we can let C to be $(\prod A_n)^*$.

Lemma 3.1. *For every $d, \delta > 0$ and for every positive integer n there exist a compact set $K_n \subset [0, 2]$ and some $\varepsilon_n > 0$ so that $\mathcal{H}^d_\infty(K_n) < \delta$ holds and that for every set $B \subset [0, 1]$ with $|B| = n + 1$ there exists some $t \in \mathbb{R}$ for which $B + [t, t + \varepsilon_n] \subset K_n$ holds.*

Proof. We prove this lemma by induction on n. We start with the case $n = 1$.

Let $a, b > 0$, let

$$K_1 = [0, a] \cup \big([0, 2] \cap \cup_{i\in\mathbb{Z}}[ia, ia + b]\big).$$

It is easy to check that if $B \subset [0, 1]$, $|B| = 2$, then there exists some $t \in \mathbb{R}$ so that $B + t \subset K_1$. Moreover, we have some freedom in choosing t; that is, there exists some $\varepsilon_1 > 0$ independent of B such that $B + [t, t + \varepsilon_1] \subset K_1$ for some t (in fact, ε_1 can be chosen to be b). The set K_1 can be covered by an interval of length a and at most $2/a + 1$ intervals of length b, thus $\mathcal{H}^d_\infty(K_1) \le a^d + (2/a + 1)b^d$. Therefore if we choose a small enough, and then b small enough, $\mathcal{H}^d_\infty(K_1)$ can be arbitrarily small.

Now let us suppose that the lemma holds for some n, we prove it for $n + 1$. So if we want a set K_{n+1} with $\mathcal{H}^d_\infty(K_n) < \delta$ for some d and δ, let us start with choosing a set K_n with $\mathcal{H}^d_\infty(K_n) < \delta/2$ and an $\varepsilon_n > 0$ so that for every set $B \subset [0, 1]$ with $|B| = n + 1$ we have $B + [t, t + \varepsilon_n] \subset K_n$ for some t.

Now let $0 < b < \varepsilon_n/2$ be so small that $(2/\varepsilon_n + 1)(2b)^d < \delta/2$. Let

$$K_{n+1} = K_n \cup \big([0, 2] \cap \cup_{i\in\mathbb{Z}}[i\varepsilon_n - b, i\varepsilon_n + b]\big).$$

Then we have $\mathcal{H}^d_\infty(K_{n+1}) < \delta$.

Let $B \subset [0, 1]$ and $|B| = n + 2$. We show that B can be translated into the set K_{n+1}. First let x be some element of B which is not the smallest or the largest one. Let $B' = B \setminus \{x\}$. Then there exists some t' so that $B' + [t', t' + \varepsilon_n] \subset K_n$. It is easy to check that there exists some t so that $[t, t+b] \subset [t', t' + \varepsilon_n]$ and $x + [t, t+b] \subset K_{n+1}$. Then clearly $B + [t, t + b] \subset K_{n+1}$. Thus ε_{n+1} can be chosen to be b. □

Let us fix some K_1 and ε_1 (so that the statement of Lemma 3.1 holds) with $\mathcal{H}^1_\infty(K_1) < 1$. Then let us fix some K_2 and ε_2 so that $\mathcal{H}^{1/2}_\infty(K_2) < 100^{-1}\varepsilon_1$. If $K_1, \dots, K_n$ and $\varepsilon_1, \dots, \varepsilon_n$ are already fixed, then let us fix K_{n+1} and ε_{n+1} so that the statement of Lemma 3.1 holds with

$$\mathcal{H}^{1/(n+1)}_\infty(K_{n+1}) < 100^{-n}\varepsilon_1\varepsilon_2 \dots \varepsilon_n. \tag{3}$$

The choice of these bounds will be clear only later.

To each of these sets K_n and ε_n we associate a set $A_n \subset \mathbb{Z}$ and some $f(n) \in \omega$ in the following way. Let $f(n)$ be a positive integer so that

$$\frac{5}{\varepsilon_n} < f(n) < \frac{10}{\varepsilon_n} \tag{4}$$

holds. Let

$$A_n = \{i \in \mathbb{Z} : [\tfrac{i}{f(n)}, \tfrac{i+4}{f(n)}] \subset K_n\} \subset \{0, 1, \ldots, 2f(n) - 1\}. \tag{5}$$

(Note that this definition implies $\frac{1}{f(n)} A_n + [0, \frac{4}{f(n)}] \subset K_n$, which we will use (only) in Claim 3.4.)

Claim 3.2. *If $B_n \subset \{0, 1, \ldots, f(n) - 1\}$ and $|B_n| = n + 1$, then there exists some $u \in \mathbb{Z}$ so that $B_n + u \subset A_n$.*

Proof. By Lemma 3.1 there exists some t so that $B_n/f(n) + [t, t + \varepsilon_n] \subset K_n$, thus $B_n + [tf(n), tf(n) + \varepsilon_n f(n)] \subset f(n)K_n$. Using (4), the length of the interval $[tf(n), tf(n) + \varepsilon_n f(n)]$ is at least 5, so it has a subinterval of the form $[u, u+4]$ with integer u. Thus $B_n + [u, u+4] \subset f(n)K_n$. Then $B_n + u \subset A_n$. $\square$

Let

$$C = (\prod_{n=1}^{\infty} A_n)^* = \frac{1}{f(1)} A_1 + \frac{1}{f(1)f(2)} A_2 + \frac{1}{f(1)f(2)f(3)} A_3 + \ldots, \tag{6}$$

thus C is compact, see our remark after definition (2).

Claim 3.3. *Let $g(n) = n + 1$. For every g-slalom $S \subset \prod_{n=1}^{\infty} \{0, 1, \ldots, f(n) - 1\}$, S^* can be covered by a translate of C.*

Proof. Let $B_n \subset \{0, 1, \ldots, f(n) - 1\}$, $|B_n| \le n + 1$ be arbitrary ($n = 1, 2, \ldots$). Let S be the g-slalom $\prod_{n \ge 1} B_n$. Then by definition (2), we have

$$S^* = \frac{1}{f(1)} B_1 + \frac{1}{f(1)f(2)} B_2 + \frac{1}{f(1)f(2)f(3)} B_3 + \ldots.$$

We can choose an integer u_n such that $B_n + u_n \subset A_n$ by Claim 3.2. Then from (6),

$$S^* + \frac{u_1}{f(1)} + \frac{u_2}{f(1)f(2)} + \ldots \subset C. \qquad \square$$

In order to finish the proof of Theorem 2.1, what is left is to show the following.

Claim 3.4. *The Hausdorff dimension of C is zero.*

Proof. For $n \ge 1$ let

$$C_n = \frac{1}{f(1)f(2)\ldots f(n)} A_n + \frac{1}{f(1)f(2)\ldots f(n+1)} A_{n+1} + \ldots. \tag{7}$$

Thus $C = C_1$ and C is the union of $|A_1||A_2| \ldots |A_{n-1}|$ many translated copies of C_n. First we show that C_n lies in a small neighbourhood of the first term on the right hand side of (7). For $k \ge 1$ we have

$$\sup \frac{1}{f(n)f(n+1)\ldots f(n+k)} A_{n+k} \le \frac{2}{f(n)f(n+1)\ldots f(n+k-1)} \le \frac{2}{f(n)2^{k-1}},$$

thus

$$\sup f(1)f(2)\ldots f(n-1)C_{n+1} \le \sum_{k=1}^{\infty} \frac{2}{f(n)2^{k-1}} = \frac{4}{f(n)},$$

and

$$f(1)f(2)\ldots f(n-1)C_{n+1} \subset [0, \tfrac{4}{f(n)}]. \tag{8}$$

Since (7) implies $C_n = \frac{1}{f(1)f(2)\dots f(n)} A_n + C_{n+1}$, using (8) we obtain

$$f(1)f(2)\dots f(n-1)C_n \subset \frac{1}{f(n)} A_n + [0, \tfrac{4}{f(n)}].$$

From the definition of A_n (5) we see that the right hand side is contained in K_n, thus

$$f(1)f(2)\dots f(n-1)C_n \subset K_n.$$

Hence

$$\mathcal{H}^d_\infty(C_n) \le (f(1)f(2)\dots f(n-1))^{-d}\mathcal{H}^d_\infty(K_n) \le \mathcal{H}^d_\infty(K_n).$$

As we have already mentioned, C is the union of $|A_1||A_2|\dots|A_{n-1}|$ many translated copies of C_n, therefore

$$\begin{aligned}\mathcal{H}^{1/n}_\infty(C) &\le |A_1||A_2|\dots|A_{n-1}|\mathcal{H}^{1/n}_\infty(C_n)\\ &\le |A_1||A_2|\dots|A_{n-1}|\mathcal{H}^{1/n}_\infty(K_n) \le (2f(1))(2f(2))\dots(2f(n-1))\mathcal{H}^{1/n}_\infty(K_n).\end{aligned}$$

Using (3) and the upper bound for $f(n)$ in (4) we obtain

$$\mathcal{H}^{1/n}_\infty(C) \le 20^{n-1}\frac{1}{\varepsilon_1\dots\varepsilon_{n-1}}100^{-(n-1)}\varepsilon_1\dots\varepsilon_{n-1} = 5^{-(n-1)}.$$

Thus for all $d > 0$ for all $n > 1/d$ we have $\mathcal{H}^d_\infty(C) \le \mathcal{H}^{1/n}_\infty(C) \le 5^{-(n-1)}$, therefore $\mathcal{H}^d_\infty(C) = 0$. So the Hausdorff dimension of the set C is zero. □

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