

L-THEORY OF C^* -ALGEBRAS

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ABSTRACT. We establish a formula for the L-theory spectrum of real C^* -algebras from which we deduce a presentation of the L-groups in terms of the topological K-groups, extending all previously known results of this kind. Along the way, we extend the integral comparison map $\tau: k \rightarrow L$ obtained in previous work by the first two authors to real C^* -algebras and interpret it using topological Grothendieck–Witt theory. Finally, we use our results to give an integral comparison between the Baum–Connes conjecture and the L-theoretic Farrell–Jones conjecture, and discuss our comparison map τ in terms of the signature operator on oriented manifolds.

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1. INTRODUCTION

This paper is concerned with certain invariants of *real* C^* -algebras. A classical and powerful invariant of real C^* -algebras is topological K-theory. However, any C^* -algebra is a ring with involution and as such also has an associated algebraic L-theory. The relation between these two invariants has been an object of investigation for a long time and the purpose of this paper is to give a definite treatment of this relation.

One of the prominent results in this direction is a theorem due to Karoubi, Miller, and Rosenberg [Kar80, Mil98, Ros95] which states that for complex C^* -algebras, there are natural isomorphisms

$$(1.1) \quad K_n(A) \cong L_n(A)$$

for all integers n . It is, however, well-known that the spectra $K(A)$ and $L(A)$ are *not* equivalent and that the isomorphism (1.1) does not hold true for real C^* -algebras A . In previous work of the first two authors, the relation between the topological K- and L-spectra of complex C^* -algebras was studied [LN18]. Neglecting 2-torsion, or more precisely after inverting 2, this relation extends to real C^* -algebras and one can summarise the situation as follows: On complex C^* -algebras, there is a unique lax symmetric monoidal natural transformation $\tau: k \rightarrow L$ which induces an equivalence

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$K[\frac{1}{2}] \rightarrow L[\frac{1}{2}]$, and this latter equivalence extends in a compatible way to real C^* -algebras. Here, $k(A)$ denotes the connective topological K-theory spectrum of A , i.e. the connective cover of $K(A)$. The map $\tau_A: k(A) \rightarrow L(A)$ induces an isomorphism on π_0 and π_1 , so that by 2-periodicity of the two theories, one finds that all L-groups are isomorphic to the corresponding topological K-groups. However, under this isomorphism the map $\tau_{\mathbb{C}}: ku \rightarrow L(\mathbb{C})$ induces multiplication by 2 on π_2 , so integrally, the periodicity in K-theory does not match up with the periodicity in L-theory. Explicitly left open in [LN18] was an integral comparison between K- and L-theory for real C^* -algebras, a gap which will be reconciled in this paper.

For the rest of this paper, C^* -algebras are now agreed to be real C^* -algebras, we will add the adjectival complex when we need it. The purpose of this paper is to explain in full generality how to describe the L-theory of C^* -algebras in terms of their topological K-theory and in particular how to express the L-groups in terms of topological K-groups. We note that both K- and L-theory of a complex C^* -algebra depend only on the underlying (real) C^* -algebra, so the case of complex C^* -algebras is treated implicitly. The following are our main results.

Theorem A. *There is a unique lax symmetric monoidal transformation $\tau: k \rightarrow L$ and the induced map*

$$k(A) \otimes_{\text{ko}} L(\mathbb{R}) \longrightarrow L(A)$$

is an equivalence for each C^ -algebra A .*

Theorem B. *Let A be a C^* -algebra. There are natural isomorphisms*

- (1) $L_0(A) \cong K_0(A)$,
- (2) $L_1(A) \cong \text{coker}(K_0(A) \xrightarrow{\eta} K_1(A))$
- (3) $L_2(A) \cong \text{ker}(K_6(A) \xrightarrow{\eta} K_7(A))$
- (4) $L_3(A) \cong K_7(A)$.

Here η is the non-trivial element in $K_1(\mathbb{R}) = \pi_1(\text{KO})$.

Remark. For a C^* -algebra A , there is the generalised Wood exact sequence

$$\cdots \longrightarrow K_{n-1}(A) \xrightarrow{\eta} K_n(A) \xrightarrow{c} K_n(A_{\mathbb{C}}) \xrightarrow{u\beta_{\mathbb{C}}^{-1}} K_{n-2}(A) \xrightarrow{\eta} K_{n-1}(A) \longrightarrow \cdots$$

Consequently, we also find canonical isomorphisms

- (1) $L_1(A) \cong \text{ker}(K_{-1}(A_{\mathbb{C}}) \xrightarrow{u} K_{-1}(A))$, and
- (2) $L_2(A) \cong \text{coker}(K_0(A) \xrightarrow{c} K_0(A_{\mathbb{C}}))$.

By the 4-fold periodicity of L-theory, Theorem B gives a natural description of all L-groups in terms of topological K-groups, see Theorem 4.5 for a more canonical formulation. Under these isomorphisms we also describe the effect on homotopy groups of the map $\tau_A: K_n(A) \rightarrow L_n(A)$ from Theorem A for $n \geq 0$ (see Proposition 5.1) as well as the exterior multiplication maps on the L-groups in terms of the exterior multiplication maps on the K-groups (see Proposition 5.3). In Section 6 we discuss a number of examples and calculate L-groups using Theorem B.

Remark. We note that Theorem A implies that two real C^* -algebras A_0 and A_1 whose K-theory spectra are equivalent as module spectra over ko have equivalent L-theory spectra. Likewise, Theorem B implies that if the K-groups of A_0 and A_1 are isomorphic as graded $\mathbb{Z}[\eta]$ -modules, where $|\eta| = 1$ and η acts in the canonical way on the K-groups, then also the L-groups of A_0 and A_1 are isomorphic.

Kasparov's KK-theory is a central tool for studying the K-theory of C^* -algebras. It therefore comes as no surprise that one would also like to study L-theory of C^* -algebras by KK-theoretic means. In the case of complex C^* -algebras, this was done in [LN18] but at the time of writing [LN18], it was not known whether L-theory of C^* -algebras is KK-invariant in general. Even the fact that it is KK-invariant on complex C^* -algebras is a result which we still find quite surprising, since KK-theory is an intrinsically analytic theory, whereas L-theory depends only on the underlying algebraic structure of a C^* -algebra. Even more, L-theory commutes with filtered

colimits of involutive rings and thus only depends on the underlying algebraic structure of proper involutive sub algebras, which do not themselves need to be C^* -algebras.

It is an immediate consequence of Theorem A that L-theory is a KK-invariant functor on C^* -algebras. Our proof works the other way around though: instead of deducing KK-invariance from Theorem A, we use it as an input for the proof of Theorem A, and we give an argument for KK-invariance based on a description for 2-complete L-theory instead. Indeed, $L[\frac{1}{2}]$ was shown to be KK-invariant in [LN18] so it remains only to see that $L(-)_{\hat{2}}$ is KK-invariant. This is a direct consequence of the following result, which we derive from [KSW16, Theorem D.1].

Theorem C. *For every Banach algebra with involution, the canonical map $L(A) \rightarrow k(A)^{tC_2}$ is a 2-adic equivalence.*

This is a topological version of Thomason’s homotopy limit problem in hermitian algebraic K-theory. This algebraic homotopy limit problem has been studied extensively, see e.g. [HKO11, BH20, BKSØ15, CDH⁺20c] for the case of fields, schemes over $\mathbb{Z}[\frac{1}{2}]$, and Dedekind rings.

Assembly Maps. As in [LN18], a major motivation for studying the relation between K- and L-theory of C^* -algebras is to obtain a precise relationship between the Baum–Connes conjecture and the L-theoretic Farrell–Jones conjecture, inspired by the observation that both of these conjectures imply the Novikov conjecture. In [LN18] such a relation was understood after inverting 2 and we offer here the following integral refinement:

Theorem D. *The map $\tau: k \rightarrow L$ induces a commutative diagram*

$$\begin{array}{ccc} \mathrm{ko}_*^G(\underline{EG}) & \xrightarrow{\mathrm{BC}} & k_*(C_r^*G) \\ \downarrow \tau & & \downarrow \tau \\ L\mathbb{R}_*^G(\underline{EG}) & \xrightarrow{\mathrm{FJ}} L_*(\mathbb{R}G) & \longrightarrow L_*(C_r^*G) \end{array}$$

where BC and FJ denote the Baum–Connes and Farrell–Jones assembly maps.

After inverting the Bott element $\beta_{\mathbb{R}}$ and 2, one recovers [LN18, Theorem D]. In addition, the kernel and cokernels of the vertical maps can in principle be described using our identification of τ on homotopy groups. For the left hand vertical map this is most effective in the case where G is torsion free as explained in Section 7.

Before our work [LN18], there have already been made several fruitful efforts to relate the surgery theoretic and the analytic approach to the Novikov conjecture, most notably the work of Higson and Roe [HR05a, HR05b, HR05c]. There, a central idea is to consider the signature operator D_M of an oriented manifold M as an appropriate K-theory class and use this to construct a comparison map from the surgery exact sequence to a 2-inverted exact sequence of topological K-groups. It has been known for a long time that the signature operator of an oriented manifold, unlike the spin Dirac operator, does not give rise to a map of spectra $\mathrm{MSO} \rightarrow \mathrm{ko}$, due to factors of 2 appearing for the signature operator on a boundary¹, see [RW06, Remark 4]. The following theorem expresses the fact that, with appropriate modifications, the signature operator does give rise to an \mathbb{E}_{∞} map $\mathrm{MSO} \rightarrow \mathrm{ko}[\frac{1}{2}]$ and clarifies its relation with the Sullivan–Ranicki orientation; a version of this theorem discarding \mathbb{E}_{∞} -structures was discussed in [RW06].

Theorem E. *The association $M^{2n} \mapsto 2^{-\lfloor n/2 \rfloor} \cdot [D_M]$ refines uniquely to a map of \mathbb{E}_{∞} -ring spectra $\mathcal{L}_{AS}: \mathrm{MSO} \rightarrow \mathrm{ko}[\frac{1}{2}]$. This map participates in the following commutative diagram of \mathbb{E}_{∞} -rings*

$$\begin{array}{ccc} \mathrm{MSO} & \xrightarrow{\mathcal{L}_{AS}} & \mathrm{ko}[\frac{1}{2}] \\ \downarrow \sigma_{\mathbb{R}} & & \downarrow \tau \\ L(\mathbb{R}) & \xrightarrow{\mathrm{can}} & L(\mathbb{R})[\frac{1}{2}] \end{array}$$

¹Also, any map between those spectra induces the trivial map on homotopy groups as follows from the fact that MSO at primes 2 vanishes $K(1)$ -locally.

where $\sigma_{\mathbb{R}}$ is the Sullivan–Ranicki orientation.

Here, the map \mathcal{L}_{AS} induces on homotopy groups a version of the L-genus, precisely the version of the L-genus that has been employed by Atiyah and Singer in their index-theoretic proof of Hirzebruch’s signature theorem [AS68]. The map $\sigma_{\mathbb{R}}$ on the other hand induces Hirzebruch’s original L-genus. Thus the result says that the two differ exactly by our comparison map.

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2. PRELIMINARIES

In this section we will briefly recall the notions of C^* -algebras, KK-theory, and L-theory.

C^* -algebras. A nice reference for C^* -algebras over \mathbb{R} and their K-theory is Schroeder’s book [Sch93], further references include [Con98, Goo82, Li03] and [Tak02, Tak03a, Tak03b] for complex C^* -algebras.

Definition 2.1. A C^* -algebra is Banach algebra A equipped with an involution $(-)^*: A \rightarrow A^{\text{op}}$ with $x^{**} = x$ such that the following two conditions hold:

- (1) $\|x^*x\| = \|x\|^2$, and
- (2) $1 + x^*x$ is invertible for all $x \in A$.

A complex C^* -algebra is a complex Banach algebra A whose underlying real Banach algebra is a C^* -algebra and where the involution is complex sesquilinear, i.e. $(\lambda x)^* = \bar{\lambda}x^*$.

Remark 2.2. The condition that $1 + x^*x$ is invertible might be a bit surprising at first glance. We note that it is a consequence of spectral calculus that this condition is automatically fulfilled for complex C^* -algebras, see e.g. [Tak02]. In the real case it can however not be left away, since for example \mathbb{C} equipped with the identity involution satisfies the other conditions but $1 + i^2 = 0$ is not invertible.

Remark 2.3. The well-known structure theorems for complex C^* -algebras have the following real analogues:

- (1) Every C^* -algebra has a faithful representation on a real Hilbert space \mathcal{H} , i.e. is isometrically isomorphic to a $*$ - and norm-closed sub algebra of $\mathcal{B}(\mathcal{H})$.
- (2) Every commutative and unital C^* -algebra is isometrically isomorphic to the C^* -algebra of C_2 -equivariant continuous functions $X \rightarrow \mathbb{C}$ for a compact Hausdorff space X equipped with a C_2 -action, where C_2 acts on \mathbb{C} by complex conjugation.

Remark 2.4. A number of remarks are in order.

- (1) Together with $*$ -homomorphisms, C^* -algebras form a category $C^*\text{Alg}$, and likewise complex C^* -algebras form a category $C^*\text{Alg}_{\mathbb{C}}$. We emphasise that C^* -algebras are not assumed to be unital, nor that $*$ -homomorphisms are assumed to preserve a unit if it exists. Requiring, however, algebras to have a unit and morphisms to preserve units, one obtains similarly the categories $C^*\text{Alg}^+$ and $C^*\text{Alg}_{\mathbb{C}}^+$ of unital C^* -algebras.
- (2) We note that $*$ -homomorphisms are automatically contractive and hence continuous.
- (3) By construction, there is a forgetful functor $C^*\text{Alg}_{\mathbb{C}} \rightarrow C^*\text{Alg}$ which we call the *realification*, moreover, the construction $A \mapsto A_{\mathbb{C}} \stackrel{\text{def}}{=} A \otimes_{\mathbb{R}} \mathbb{C}$ extends to a natural functor $C^*\text{Alg} \rightarrow C^*\text{Alg}_{\mathbb{C}}$, which we call the *complexification*.
- (4) There are unitalisation functors $C^*\text{Alg} \rightarrow C^*\text{Alg}^+$ and $C^*\text{Alg}_{\mathbb{C}} \rightarrow C^*\text{Alg}_{\mathbb{C}}^+$ which come with natural split exact sequences

$$0 \longrightarrow A \longrightarrow A^+ \longrightarrow \mathbb{R} \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow B \longrightarrow B^+ \longrightarrow \mathbb{C} \longrightarrow 0$$

respectively. If A is unital, then A^+ is canonically isomorphic to $A \times \mathbb{R}$, and likewise in the complex case.

- (5) The complexification functor is compatible with unitalisation, whereas the realification functor is not compatible with unitalisation. More precisely the solid diagram commutes, whereas the diagram involving dashed arrows does not.

$$\begin{array}{ccc} C^* \text{Alg} & \longrightarrow & C^* \text{Alg}^+ \\ \begin{array}{c} \nearrow \\ \downarrow \\ \dashrightarrow \end{array} & & \begin{array}{c} \nearrow \\ \downarrow \\ \dashrightarrow \end{array} \\ C^* \text{Alg}_{\mathbb{C}} & \longrightarrow & C^* \text{Alg}_{\mathbb{C}}^+ \end{array}$$

- (6) The categories $C^* \text{Alg}$ and $C^* \text{Alg}_{\mathbb{C}}$ are each equipped with a canonical symmetric monoidal structure, the maximal tensor product over \mathbb{R} and \mathbb{C} , respectively. The maximal tensor product preserves short exact sequences of C^* -algebras and topological K-theory is canonically lax symmetric monoidal.

Definition 2.5. A C^* -algebra is called separable if it contains a countable and dense subset. The full subcategory of $C^* \text{Alg}_{(\mathbb{C})}$ on separable C^* -algebras will be written $C^* \text{Alg}_{(\mathbb{C})}^{\text{sep}}$.

The complexification and realification functors restrict to the subcategory of separable algebras. In addition, we note that every C^* -algebra is the union of its separable sub C^* -algebras and that the collection of separable sub C^* -algebras forms a filtered poset. For technical reasons, we will restrict our attention to separable algebras momentarily. However, all invariants F of C^* -algebras we shall consider (i.e. topological K-theory and L-theory) send an algebra A to the filtered colimit of F applied to the separable sub algebras of A , and consequently, we can get rid of the separability assumptions.

2.1. KK-theory. In his seminal work on the Novikov conjecture [Kas88], Kasparov invented (equivariant) bivariant topological K-theory, known as KK-theory. Phrased in categorical language, Kasparov's machine allowed to construct a tensor triangulated category KK and a functor

$$C^* \text{Alg}^{\text{sep}} \longrightarrow \text{KK}$$

which was later shown to be a localisation (necessarily at the KK-equivalences, i.e. those $*$ -homomorphisms whose induced map in the KK-category is an isomorphism) [Cun87] and to be the initial functor to an additive category which is split exact and stable [Hig87], see e.g. [BEL21a] for more precise statements and a guide through (parts of) the literature. In [LN18], it was then observed that the ∞ -categorical localisation of $C^* \text{Alg}^{\text{sep}}$ at the KK-equivalences is a stably symmetric monoidal ∞ -category whose homotopy category is canonically equivalent to the tensor triangulated category KK of Kasparov. This observation has also been taken up in [BEL21a] (including extensions of these results to possibly non-separable C^* -algebras) in the equivariant case and was used in [BEL21b] in a proof of an equivariant form of Paschke duality.

Definition 2.6. We denote by $\text{KK} = C^* \text{Alg}^{\text{sep}}[\text{KK}^{-1}]$ the ∞ -categorical localisation of $C^* \text{Alg}^{\text{sep}}$ at the KK-equivalences. Likewise, we denote by $\text{KK}_{\mathbb{C}} = C^* \text{Alg}_{\mathbb{C}}^{\text{sep}}[\text{KK}^{-1}]$ the variant for complex C^* -algebras.

Remark 2.7. In [LN18], different notation was used: In loc. cit. the authors were focussed mostly on the complex case and therefore denoted $C^* \text{Alg}_{\mathbb{C}}^{\text{sep}}[\text{KK}^{-1}]$ by KK_{∞} , and its real variant by $\text{KK}_{\infty}^{\mathbb{R}}$; the subscript ∞ was added to make clear that one was now working with an appropriate ∞ -category rather than a triangulated category. We refrain from adding this subscript in this paper, however.

Definition 2.8. The topological K-theory functor for separable C^* -algebras is given by the composite

$$\text{K}: C^* \text{Alg}^{\text{sep}} \longrightarrow \text{KK} \longrightarrow \text{Sp}$$

where the first functor is the localisation functor and the second is the corepresented functor $\text{map}_{\text{KK}}(\mathbb{R}, -)$.

Remark 2.9. There are of course other, more classical definitions of topological K-theory functors [Joa04, Joa03], and it was shown in [LN18] that they are canonically equivalent to the definition given above. These more classical definitions are in fact given for possibly non-separable algebras and satisfy

$$K(A) \simeq \operatorname{colim}_{A' \subseteq_{\text{sep}} A} K(A')$$

so we may also view the above definition as describing K-theory of possibly non-separable algebras. In [BEL21a] this was formalised by considering the ind-completion of KK and again considering the functor corepresented by \mathbb{R} .

This definition, however, does not give all structure that topological K-theory has: For instance, it is a purely formal consequence of the definitions that K-theory sends certain short exact sequences (e.g. where the surjection is a Schochet fibration or admits a cpc split) to fibre sequences, but it is not a priori clear that it sends all short exact sequences of C^* -algebras to fibre sequences. However, this is known to be true, see e.g. [BEL21a, Theorem 1.14 for $X = *$] for a list of further properties of the K-theory functor.

2.2. L-theory. In this subsection, we review some basic properties of L-theory which we will use throughout this paper, see also [LN18, §2.2] for a further summary.

For our purposes, L-theory is most naturally considered as a functor introduced by Ranicki in [Ran92]

$$\operatorname{Ring}^{\text{inv}} \longrightarrow \operatorname{Sp}$$

where $\operatorname{Ring}^{\text{inv}}$ is the category of involutive rings with ring homomorphisms preserving the involution. In fact, this functor can be written as the composition

$$\operatorname{Ring}^{\text{inv}} \longrightarrow \operatorname{Cat}_{\infty}^{\text{P}} \longrightarrow \operatorname{Sp}$$

where $\operatorname{Cat}_{\infty}^{\text{P}}$ is the ∞ -category of Poincaré categories on which L-theory is a natural invariant, see [CDH⁺20a, CDH⁺20b, CDH⁺20c] for applications of this formalism to Grothendieck–Witt theory of number rings. Together with [CDH⁺22], or using results of Laures–McClure [LM14, LM21], L-theory is canonically endowed with a lax symmetric monoidal structure. The first functor in the above composite sends a ring with involution R to the pair $(\mathcal{D}^p(R), \mathcal{Q}^s)$, so more precisely we are considering projective, 4-periodic symmetric L-theory of involutive rings in the sense of [Ran92]. Prior to the work [CDH⁺20a, CDH⁺20b, CDH⁺20c], the third author had introduced L-spectra for dg-categories over $\mathbb{Z}[\frac{1}{2}]$ with weak equivalences [Sch17, §7]. In [CDH⁺20b, Appendix B.2], it is shown that for $\mathbb{Z}[\frac{1}{2}]$ -algebras with involution, the two constructions of L-spectra are naturally equivalent.

There are natural forgetful functors

$$C^* \operatorname{Alg}^+ \longrightarrow \operatorname{Ring}_{\mathbb{Z}[\frac{1}{2}]}^{\text{inv}} \longrightarrow \operatorname{Ring}^{\text{inv}}$$

which define L-theory of *unital* C^* -algebras. Since many of the possibly different notions of L-theory agree on rings in which 2 is invertible, and since this paper is concerned with C^* -algebras, we shall from now on restrict our attention to $\mathbb{Z}[\frac{1}{2}]$ -algebras with involution as the domain of L-theory.

L-theory for non-unital algebras. For our applications, which involve KK-theory, it is necessary to define L-theory for possibly non-unital algebras. For this we define a unitalisation of non-unital rings in the usual way

$$\operatorname{Ring}_{\mathbb{Z}[\frac{1}{2}], \text{nu}}^{\text{inv}} \longrightarrow \operatorname{Ring}_{\mathbb{Z}[\frac{1}{2}]}^{\text{inv}} \quad R \mapsto R^+ = R \rtimes \mathbb{Z}[\frac{1}{2}]$$

and note again that for unital $\mathbb{Z}[\frac{1}{2}]$ -algebras S , we have $S^+ \cong S \times \mathbb{Z}[\frac{1}{2}]$. We then define L-theory on non-unital $\mathbb{Z}[\frac{1}{2}]$ -algebras as follows

$$L(R) \stackrel{\text{def}}{=} \operatorname{fib}(L(R^+) \rightarrow L(\mathbb{Z}[\frac{1}{2}])).$$

Since L-theory commutes with finite products [LN18, Corollary 4.4] (for this to be true it is crucial to work with *projective* L-theory, rather than free L-theory which appears in the h-cobordism

classification program in surgery theory), we have not changed the definition of L-theory on unital rings, up to canonical equivalence.

However, from the point of view of applying L-theory to C^* -algebras, we now have constructed two functors

$$C^* \text{ Alg} \longrightarrow \text{Ring}^{\text{inv}}$$

one given by $A \mapsto A^+ = A \rtimes \mathbb{R}$ and the other one given by $A \mapsto A^+ = A \rtimes \mathbb{Z}[\frac{1}{2}]$, i.e. we can either unitalise in \mathbb{R} -algebras or in $\mathbb{Z}[\frac{1}{2}]$ -algebras. Moreover, for complex algebras, we have three such functors, by adjoining a unit in \mathbb{C} -algebras, \mathbb{R} -algebras, or $\mathbb{Z}[\frac{1}{2}]$ -algebras, respectively². We note that for a C^* -algebra A , there is a natural pullback diagram

$$\begin{array}{ccccc} A \rtimes \mathbb{Z}[\frac{1}{2}] & \longrightarrow & A \rtimes \mathbb{R} & \longrightarrow & A \rtimes \mathbb{C} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z}[\frac{1}{2}] & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{C} \end{array}$$

where the right most vertical part only exists if A is a complex C^* -algebra and where the vertical maps are split surjective. It is a theorem of Ranicki's [Ran81], see e.g. [LN18, Corollary 4.3] that both squares induce pullback squares on L-theory. Consequently, extending L-theory to non-unital (complex) C^* -algebras can be performed either by adjoining a unit in \mathbb{C} -algebras, or by forgetting to the underlying real C^* -algebra and then adjoining a unit in \mathbb{R} -algebras, or by forgetting to the underlying $\mathbb{Z}[\frac{1}{2}]$ -algebra and adjoining a unit there.

Remark 2.10. We do not expect the diagram

$$\begin{array}{ccc} L(A \rtimes \mathbb{Z}) & \longrightarrow & L(A \rtimes \mathbb{Z}[\frac{1}{2}]) \\ \downarrow & & \downarrow \\ L(\mathbb{Z}) & \longrightarrow & L(\mathbb{Z}[\frac{1}{2}]) \end{array}$$

to be a pullback for every $\mathbb{Z}[\frac{1}{2}]$ -algebra A . As a consequence, we do not expect the definition of L-theory for non-unital rings to be independent of the base over which the unitalisation is performed in general. Ranicki however shows that this square is a pullback if symmetric L-theory is replaced by quadratic L-theory, see [Ran81, 6.3.1].

Finally, as explained in [LN18, Appendix], the fact that L-theory is lax symmetric monoidal on unital C^* -algebras allows to deduce that L-theory as defined above is in fact canonically lax symmetric monoidal on all C^* -algebras.

3. PROOF OF THEOREM C

For convenience we state again the theorem we shall prove in this section. We emphasise that KK-theory is not used in this proof.

Theorem 3.1. *Let A be a Banach algebra with involution. Then the canonical map $L(A) \rightarrow k(A)^{tC_2}$ is a 2-adic equivalence.*

In the proof of this theorem and in fact also of Theorem A, we will make use of the topological Grothendieck–Witt spectra introduced in [Sch17, §10] which we denote by $\text{GW}_{\text{top}}(A)$. We recall that these are defined as the geometric realisation

$$\text{GW}_{\text{top}}(A) = \text{colim}_{n \in \Delta^{\text{op}}} \text{GW}(C^0(\Delta^n, A))$$

where $C^0(\Delta^n, A)$ is the ring of continuous A -valued functions on the n -simplex Δ^n , equipped with pointwise involution, and GW is its (algebraic) Grothendieck–Witt spectrum. We recall that connective topological K-theory $k(A)$ admits a similar description in terms of connective algebraic K-theory K_{alg} :

$$k(A) = \text{colim}_{n \in \Delta^{\text{op}}} K_{\text{alg}}(C^0(\Delta^n, A)).$$

²Of course, one could also unitalise in \mathbb{Z} -algebras, but see Remark 2.10 below.

For any ring with involution R with $2 \in R^\times$ there is a natural fibre sequence

$$(\mathrm{K}_{\mathrm{alg}}(R))_{hC_2} \longrightarrow \mathrm{GW}(R) \longrightarrow \mathrm{L}(R),$$

see e.g. [CDH⁺20b, Main Theorem] or [Sch17, Theorem 7.6] using that the Grothendieck–Witt spectra of [CDH⁺20b] and of [Sch17] agree, see [CDH⁺20b, Appendix B.2]. More specifically, in the notation of [CDH⁺20b], $\mathrm{GW}(R)$ is given by $\mathrm{GW}(R; \mathcal{Q}_R^s)$, and likewise $\mathrm{L}(R)$ is given by $\mathrm{L}(R; \mathcal{Q}_R^s)$; we remark here that by assumption 2 is invertible in R , many of the a priori different versions of Grothendieck–Witt theory studied in [CDH⁺20b, CDH⁺20c] collapse to the same object, which we here simply denote by $\mathrm{GW}(R)$. Equivalently this fibre sequence can be written as the following natural pullback diagram.

$$(3.1) \quad \begin{array}{ccc} \mathrm{GW}(R) & \longrightarrow & \mathrm{L}(R) \\ \downarrow & & \downarrow \\ \mathrm{K}_{\mathrm{alg}}(R)^{hC_2} & \longrightarrow & \mathrm{K}_{\mathrm{alg}}(R)^{tC_2} \end{array}$$

We may then likewise define

$$\mathrm{L}_{\mathrm{top}}(A) = \operatorname{colim}_{n \in \Delta^{\mathrm{op}}} \mathrm{L}(C^0(\Delta^n, A)).$$

An astonishing feature of algebraic L-theory is the following homotopy invariance statement, see e.g. [Sch17, Remark 10.4].

Proposition 3.2. *For every Banach algebra with involution, the canonical map $\mathrm{L}(A) \rightarrow \mathrm{L}_{\mathrm{top}}(A)$ is an equivalence.*

Proof of Theorem C. We first claim that there is the following natural square of spectra

$$\begin{array}{ccc} \mathrm{GW}_{\mathrm{top}}(A) & \longrightarrow & \mathrm{L}(A) \\ \downarrow & & \downarrow \\ \mathrm{k}(A)^{hC_2} & \longrightarrow & \mathrm{k}(A)^{tC_2} \end{array}$$

and that this square is a pullback square. To see this we use Proposition 3.2 to replace $\mathrm{L}(A)$ by its topological variant and by taking the geometric realization of (3.1) we get a diagram as desired (using the canonical colimit interchange map for the lower two corners). To see that it is a pullback we use that the canonical map induced on horizontal fibres is an equivalence, since homotopy orbits commute with the geometric realization. We wish to show that the right vertical map is a 2-adic equivalence. It therefore suffices to show that the left vertical map is a 2-adic equivalence. To see this, one considers the Bott-periodic analog $\mathbb{G}\mathrm{W}_{\mathrm{top}}(A)$ of $\mathrm{GW}_{\mathrm{top}}(A)$ also used in [KSW16, Appendix D]. It participates in a commutative square

$$\begin{array}{ccc} \mathrm{GW}_{\mathrm{top}}(A) & \longrightarrow & \mathbb{G}\mathrm{W}_{\mathrm{top}}(A) \\ \downarrow & & \downarrow \\ \mathrm{k}(A)^{hC_2} & \longrightarrow & \mathrm{K}(A)^{hC_2} \end{array}$$

where $\mathrm{K}(A)$ denotes periodic topological K-theory of A . We now show that this diagram is a pullback diagram; the argument is similar to [Sch17, Theorem 8.14]. Indeed, the horizontal fibres are coconnected spectra, see [KSW16, Proof of Theorem D.1] for the top horizontal fibre, whereas the vertical fibres are 4-periodic, as is the map between them, compare [Sch17, Remark 7.7]. The total homotopy fibre of the above square is therefore both periodic and bounded above and thus trivial. Now, finally, by [KSW16, Theorem D.1] the right vertical map in the above square is a 2-adic equivalence. Theorem C is therefore proven. \square

Corollary 3.3. *The functor L descends to a functor $\mathrm{KK} \rightarrow \mathrm{Sp}$.*

Proof. The arithmetic fracture square provides a pullback square of functors

$$\begin{array}{ccc} L & \longrightarrow & L_2^\wedge \\ \downarrow & & \downarrow \\ L[\frac{1}{2}] & \longrightarrow & L_2^\wedge[\frac{1}{2}] \end{array}$$

so it suffices to show that $L[\frac{1}{2}]$ and L_2^\wedge are KK-invariant. The former is a direct consequence of the natural isomorphism between $K_n(-)[\frac{1}{2}]$ and $L_n(-)[\frac{1}{2}]$ (see also [LN18]) and the latter follows from Theorem C. \square

Corollary 3.4. *The functor $\tau_{\geq 0}\mathrm{GW}_{\mathrm{top}}: \mathrm{KK} \rightarrow \mathrm{Sp}_{\geq 0}$ is canonically equivalent to $\mathfrak{k} \oplus \mathfrak{k}$.*

Proof. From the fibre sequence

$$\mathfrak{k}_{hC_2} \longrightarrow \mathrm{GW}_{\mathrm{top}} \longrightarrow L$$

and Corollary 3.3, we deduce that $\mathrm{GW}_{\mathrm{top}}$ is a KK-invariant functor. In addition, we now show that the induced functor $\tau_{\geq 0}\mathrm{GW}_{\mathrm{top}}: \mathrm{KK} \rightarrow \mathrm{Sp}_{\geq 0}$ is excisive, i.e. sends pushout diagrams to pullback diagrams. For this, we consider again the pullback diagram

$$\begin{array}{ccc} \tau_{\geq 0}\mathrm{GW}_{\mathrm{top}} & \longrightarrow & \tau_{\geq 0}L \\ \downarrow & & \downarrow \\ \tau_{\geq 0}\mathfrak{k}^{hC_2} & \longrightarrow & \tau_{\geq 0}\mathfrak{k}^{tC_2} \end{array}$$

of functors taking values in connective spectra. We wish to show that, as such $\tau_{\geq 0}\mathrm{GW}_{\mathrm{top}}$ is excisive. Note that this means that pullbacks are taken in connective spectra, so being excisive as connective spectrum valued functor is not the same as being excisive when viewed as a spectrum valued functor (via the canonical inclusion of connective spectra in all spectra). We now observe that $\tau_{\geq 0}\mathfrak{k}^{hC_2}$ is excisive, so it suffices to show that the fibre of the right vertical map is excisive as well. This follows from the formula for the failure of excisiveness of L-theory obtained by Ranicki, see [LN18, §4.1] for the relevant version of Ranicki's result.

We now observe that $\pi_0(\mathrm{GW}_{\mathrm{top}}(A))$ is naturally isomorphic to $\pi_0(\mathfrak{k}(A) \oplus \mathfrak{k}(A))$, induced by sending a tuple (P_1, P_2) of projective modules over A to the hermitian form which is the canonical positive definite form on P_1 and the canonical negative definite form on P_2 , see [Kar80, Theorem 2.3]. Since \mathfrak{k} is corepresented by \mathbb{R} , we obtain a canonical transformation $\mathfrak{k} \oplus \mathfrak{k} \rightarrow \mathrm{GW}_{\mathrm{top}}(A)$ which induces an isomorphism on π_0 . Since both sides are excisive when viewed as taking values in connective spectra, we deduce that this map induces an isomorphism in π_n for all $n \geq 0$. \square

Remark 3.5. One can also give a direct argument for a natural equivalence $\tau_{\geq 0}\mathrm{GW}_{\mathrm{top}}(A) \simeq \mathfrak{k}(A) \oplus \mathfrak{k}(A)$, see [Kar80, Theorem 2.3] for the version on homotopy groups. Informally, the map from right to left is obtained as follows: First, one shows that $\tau_{\geq 0}\mathrm{GW}_{\mathrm{top}}(A)$ is the group completion of the topological category $\mathrm{Unimod}(A)$ of unimodular hermitian forms over A , see e.g. [Sch17, Corollary A.2]. Then one shows that the functor $\mathrm{Proj}(A) \times \mathrm{Proj}(A) \rightarrow \mathrm{Unimod}(A)$, given by sending (P_1, P_2) to $(P_1 \oplus P_2, \sigma^{\mathrm{pos}} \oplus \sigma^{\mathrm{neg}})$ is an equivalence of topological categories; here, σ^{pos} denotes the canonical positive definite form on P_1 and σ^{neg} its negative definite variant. The main statement here is to see that the group of isometries of $(P, \sigma^{\mathrm{pos}})$ is homotopy equivalent to the group of isomorphisms of P ; a shadow of this fact is [Kar80, Lemma 2.9].

This perspective shows that the equivalence in fact holds more generally for C -algebras in the sense of [Kar80, Definition 2.2], but we shall not make use of this fact in this paper.

Having this equivalence, one deduces that $\tau_{\geq 0}\mathrm{GW}$ is KK-invariant. From the fibre sequence

$$\mathfrak{k}_{hC_2} \longrightarrow \tau_{\geq 0}\mathrm{GW}_{\mathrm{top}} \longrightarrow \tau_{\geq 0}L$$

it then follows that $\tau_{\geq 0}L$, and therefore by periodicity also L , is also KK-invariant. However, this perspective does not immediately give a proof of Theorem C. We have decided to deduce the description of $\mathrm{GW}_{\mathrm{top}}$ in the way presented rather than showing the equivalence $\tau_{\geq 0}\mathrm{GW}_{\mathrm{top}}(A) \simeq \mathfrak{k}(A) \oplus \mathfrak{k}(A)$ by hand, which might in fact be the more natural thing to do.

4. PROOF OF THEOREM A & B

In this section, we prove Theorem A and Theorem B from the introduction. Again, we recall the statements here for convenience. We emphasize at this point that the proofs of Theorem A and B rely only on the consequence of Theorem C that L-theory is a KK-invariant functor, not on Theorem C itself. In particular, Theorems A and B can also be derived using the argument outlined in Remark 3.5. This approach makes no use of the fact that $L[\frac{1}{2}]$ is KK-invariant, which was deduced in [LN18] from the fact that L is KK-invariant on complex C^* -algebras, which in turn was proven by using that Theorem B was known previously for complex C^* -algebras as indicated in the introduction [Kar80, Mil98, Ros05].

Theorem 4.1. *There is a unique lax symmetric monoidal transformation $\tau: \mathbf{k} \rightarrow \mathbf{L}$ and the induced map*

$$\mathbf{k}(A) \otimes_{\mathbf{ko}} \mathbf{L}(\mathbb{R}) \longrightarrow \mathbf{L}(A)$$

is an equivalence for each C^ -algebra A .*

We note that any transformation $\tau: \mathbf{k} \rightarrow \mathbf{L}$ automatically factors through the connective cover $\ell \rightarrow \mathbf{L}$. Since the canonical map $\ell(A) \otimes_{\ell(\mathbb{R})} \mathbf{L}(\mathbb{R}) \rightarrow \mathbf{L}(A)$ is an equivalence, the statement that the canonical map

$$(4.1) \quad \mathbf{k}(A) \otimes_{\mathbf{ko}} \ell(\mathbb{R}) \longrightarrow \ell(A)$$

is an equivalence directly implies Theorem 4.1. It is this statement that we shall actually prove. A priori, the map (4.1) being an equivalence is a stronger statement than Theorem 4.1 since it also implies that there is no b -torsion in $\mathbf{k}(A) \otimes_{\mathbf{ko}} \ell(\mathbb{R})$, where $b \in L_4(\mathbb{R})$ is a generator, or equivalently, that the spectrum $\mathbf{k}(A) \otimes_{\mathbf{ko}} \tau_{<0} \mathbf{L}(\mathbb{R})$ is coconnected. Using the Whitehead filtration of $\tau_{<0} \mathbf{L}(\mathbb{R})$ which has graded pieces given by $\mathbb{Z}[4k]$ for $k \leq -1$, and the presentation $\mathbb{Z} = (\mathbf{ko}/\eta)/\beta_{\mathbb{C}}$, coconnectedness can, however, also be shown directly. Thus the conclusion of Theorem 4.1 and the statement that the map (4.1) is an equivalence are in fact equivalent.

Proof of Theorem 4.1. By the results of the previous section, we know that we may view \mathbf{L} as a functor $\mathbf{KK} \rightarrow \mathbf{Sp}$. As such, it is canonically lax symmetric monoidal, because L-theory is lax symmetric monoidal on C^* -algebras. In other words, \mathbf{L} is canonically an object of $\mathbf{Alg}(\mathbf{Fun}(\mathbf{KK}, \mathbf{Sp}))$ where algebras are formed with respect to the Day convolution symmetric monoidal structure on $\mathbf{Fun}(\mathbf{KK}, \mathbf{Sp})$. As such it receives a unique algebra map from the unit, which is given by the functor $\mathbf{map}_{\mathbf{KK}}(\mathbb{R}, -) \simeq \mathbf{k}$. Therefore, as in [LN18] there is a unique lax symmetric monoidal transformation $\tau: \mathbf{k} \rightarrow \mathbf{L}$.

We now consider the cofibre sequence

$$\mathbf{k}_{hC_2} \xrightarrow{\text{hyp}} \tau_{\geq 0} \mathbf{GW}_{\text{top}} \longrightarrow \tau_{\geq 0} \mathbf{L}$$

and identify $\tau_{\geq 0} \mathbf{GW}_{\text{top}}$ with $\mathbf{k} \oplus \mathbf{k}$ using Corollary 3.4. First, we show that the C_2 -action on \mathbf{k} is trivial: Indeed, since \mathbf{k} is corepresented by \mathbb{R} , this is equivalent to the statement that the C_2 -action on $\mathbf{ko} = \mathbf{k}(\mathbb{R})$ induced from sending a projective module to its dual is trivial. This is of course classical and follows for instance from the existence of a positive definite form on each finite dimensional \mathbb{R} -vector space. Alternatively, one can also use that the space of \mathbb{E}_{∞} -self maps of \mathbf{ko} is contractible. We deduce that under the equivalence $\tau_{\geq 0} \mathbf{GW}_{\text{top}} \simeq \mathbf{k} \oplus \mathbf{k}$, the map $\text{hyp}: \mathbf{k}_{hC_2} \rightarrow \mathbf{k} \oplus \mathbf{k}$ can equivalently be described by a C_2 -equivariant map $r^*(\mathbf{k}) \rightarrow r^*(\mathbf{k} \oplus \mathbf{k})$ where $r: BC_2 \rightarrow *$ is the unique map. Therefore, the map hyp is equivalently described by a map in the category $\mathbf{Fun}(\mathbf{KK}, \mathbf{Fun}(BC_2, \mathbf{Sp})) \simeq \mathbf{Fun}(BC_2, \mathbf{Fun}(\mathbf{KK}, \mathbf{Sp}))$. The Yoneda Lemma induces the fully faithful inclusion

$$\mathbf{Fun}(BC_2, \mathbf{KK}^{\text{op}}) \longrightarrow \mathbf{Fun}(BC_2, \mathbf{Fun}(\mathbf{KK}, \mathbf{Sp}))$$

and the map hyp is a map between objects in the image. Therefore, the map hyp is uniquely determined by the C_2 -equivariant map $r^*(\mathbf{ko} \oplus \mathbf{ko}) \rightarrow r^*(\mathbf{ko})$, whose induced map $\mathbf{ko}_{hC_2} \rightarrow \mathbf{ko} \oplus \mathbf{ko} \simeq \mathbf{GW}_{\text{top}}(\mathbb{R})$ is the map $\text{hyp}(\mathbb{R})$. The same is true for the map

$$\mathbf{k}(A) \otimes_{\mathbf{ko}} \mathbf{ko}_{hC_2} \xrightarrow{\mathbf{k}(A) \otimes \text{hyp}(\mathbb{R})} \mathbf{k}(A) \otimes_{\mathbf{ko}} (\mathbf{ko} \oplus \mathbf{ko})$$

so we deduce that the map $\text{hyp}(A)$ identifies with the map $\text{id}_{k(A)} \otimes_{\text{ko}} \text{hyp}(\mathbb{R})$. Therefore, we deduce that

$$\ell(A) = \text{cofib}(\text{hyp}(A)) = k(A) \otimes_{\text{ko}} \text{cofib}(\text{hyp}(\mathbb{R})) = k(A) \otimes_{\text{ko}} \ell(\mathbb{R})$$

as claimed. To see that the map is the one we claimed, it suffices to note that the induced map

$$k(A) \longrightarrow k(A) \otimes_{\text{ko}} \ell(\mathbb{R}) \xrightarrow{\simeq} \ell(A)$$

is natural in A and for $A = \mathbb{R}$ agrees with the map $\tau_{\mathbb{R}}: \text{ko} \rightarrow \ell(\mathbb{R})$. \square

Remark 4.2. Let us consider the following commutative diagram

$$\begin{array}{ccccc} k & \xrightarrow{\Delta} & k \oplus k & \longrightarrow & k \\ \downarrow & & \downarrow \simeq & & \downarrow \hat{\tau} \\ k_{hC_2} & \xrightarrow{\text{hyp}} & \tau_{\geq 0}\text{GW}_{\text{top}} & \longrightarrow & \ell \end{array}$$

where the left vertical map is the canonical projection map and the middle vertical map is the equivalence of Remark 3.5. As a consequence, the (induced) map Δ is indeed the diagonal map. We obtain a canonical map $\hat{\tau}$ induced on horizontal cofibres. Now we may consider the composite

$$k \xrightarrow{(\text{id}, 0)} k \oplus k \xrightarrow{\simeq} \tau_{\geq 0}\text{GW}_{\text{top}}$$

considered as a map from the top right term in the above diagram. This map has the following interpretation: It arises by observing that $k(A)$ can be described as the K-theory of the topological category of positive definite forms on projective A -modules. The canonical inclusion to the category of all unimodular forms then induces the map $k \rightarrow \tau_{\geq 0}\text{GW}_{\text{top}}$ just explained. With this interpretation, one sees that this map is canonically a lax symmetric monoidal transformation. Using that also the map $\text{GW}_{\text{top}} \rightarrow L$ is lax symmetric monoidal, see [CDH⁺22] for a general statement along these lines, we find that the composite

$$k \rightarrow \tau_{\geq 0}\text{GW}_{\text{top}} \rightarrow \ell$$

on the one hand agrees with $\hat{\tau}$ (by construction) and is canonically lax symmetric monoidal. By the uniqueness part of Theorem 4.1, we deduce that $\hat{\tau} = \tau$. In particular, we deduce that ℓ is described as the cofibre of a transformation

$$\widetilde{k_{hC_2}} \longrightarrow k$$

where the tilde denotes reduced C_2 -orbits, i.e. the cofibre of the projection map $k \rightarrow k_{hC_2}$. This transformation is, similarly as before, determined by its induced map $\widetilde{\text{ko}_{hC_2}} \rightarrow \text{ko}$. A natural guess is that this map is given as follows. We recall that the C_2 -action on ko is trivial, so that the above map is equivalently described by a map $BC_2 \rightarrow \text{End}(\text{ko})$, landing in the component of the trivial map. We can then consider the canonical map

$$BC_2 \longrightarrow \text{gl}_1(\text{ko}) \subseteq \text{End}(\text{ko})$$

which lands in the component of the identity, and shift it to the component of the trivial map using the additive structure on $\text{End}(\text{ko})$. We note that precomposing this map with the canonical map $B\mathbb{Z} \rightarrow BC_2$, we obtain a map $\Sigma\text{ko} \rightarrow \text{ko}$ which is given by the multiplication by η . This would be compatible with the discussion at the end of this section, but we refrain from attempting to prove that the map $\widetilde{\text{ko}_{hC_2}} \rightarrow \text{ko}$ is indeed given by this construction.

Next, we aim to prove Theorem B from the introduction. First, we recall that $\text{KSp} = \text{K}(\mathbb{H})$ is the topological K-theory spectrum of the quaternions and denote by ksp its connective cover. As a further preparation we denote by $\tilde{\eta}: \Sigma\text{ko}/2 \rightarrow \text{ko}$ an extension of the η -multiplication $\Sigma\text{ko} \rightarrow \text{ko}$ to $\Sigma\text{ko}/2$. Such an extension exists as $2\eta = 0$, but is not unique. Regardless which extension is chosen, we have the following symplectic analogue of Wood's theorem – recall that Wood's theorem states that $\text{cofib}(\eta) = \text{ku}$.

Lemma 4.3. *There is an equivalence $\text{cofib}(\tilde{\eta}) \simeq \text{ksp}$.*

Proof. Since $\mathrm{KSp} \simeq \Sigma^4 \mathrm{KO}$, we may equivalently show that there is a fibre sequence

$$\Sigma \mathrm{KO}/2 \xrightarrow{\tilde{\eta}} \mathrm{KO} \longrightarrow \Sigma^4 \mathrm{KO}.$$

To do so, we first recall the homotopy groups of $\mathrm{KO}/2$ and then use the long exact sequence to calculate the homotopy of the cofibre $C = \mathrm{cofib}(\tilde{\eta})$. We will then show that the homotopy groups of this cofibre are, up to a shift of 4, a free KO_* -module of rank 1, giving the lemma. We have:

$$\pi_n(\mathrm{KO}/2) \cong \begin{cases} \mathbb{Z}/2 & \text{for } n \equiv 0, 1, 3, 4 \pmod{8} \\ \mathbb{Z}/4 & \text{for } n = 2 \pmod{8} \\ 0 & \text{else} \end{cases}$$

Investing the above long exact sequence we then see that the homotopy groups of C are indeed isomorphic to the ones of $\Sigma^4 \mathrm{KO}$. In addition, we note that the canonical map $\mathrm{KO} \rightarrow C$ has the following properties: It induces an isomorphism on π_0 and the multiplication by 4 map on π_4 . We will denote a generator of $\pi_4(\mathrm{KO})$ by x and note that $x^2 = 4\beta_{\mathbb{R}}$, showing that the x -multiplication of C and $\Sigma^4 \mathrm{KO}$ agree. To see that $\pi_*(C)$ is then isomorphic to $\pi_*(\Sigma^4 \mathrm{KO})$, it suffices to show that the η -multiplication gives surjections

$$\pi_4(C) \longrightarrow \pi_5(C) \longrightarrow \pi_6(C).$$

We then consider the diagram

$$\begin{array}{ccccc} \pi_4(C) & \longrightarrow & \pi_5(C) & \longrightarrow & \pi_6(C) \\ \downarrow & & \downarrow \cong & & \downarrow \cong \\ \pi_2(\mathrm{KO}/2) & \longrightarrow & \pi_3(\mathrm{KO}/2) & \longrightarrow & \pi_4(\mathrm{KO}/2) \end{array}$$

where the horizontal maps are the η -multiplications. It then suffices to verify that the η -multiplication in $\mathrm{KO}/2$ give the surjections as needed, which in turn follows from the fibre sequence

$$\Sigma \mathrm{KO}/2 \longrightarrow \mathrm{KO}/2 \longrightarrow \mathrm{KU}/2. \quad \square$$

We note that the fact that $\mathrm{KSp} \simeq \Sigma^4 \mathrm{KO}$ is, up to a shift, a free KO -module makes the symplectic Wood theorem much easier to prove than the complex Wood theorem.

Corollary 4.4. *There is a ko -linear map $\mathrm{ksp} \rightarrow \ell(\mathbb{R})$ which induces an isomorphism on π_0 and consequently an equivalence $\tau_{\leq 3} \mathrm{ksp} \simeq \tau_{\leq 3} \ell(\mathbb{R})$.*

Proof. Since the canonical map $\mathrm{ko} \rightarrow \mathrm{ksp}$ induced by the map $\mathbb{R} \rightarrow \mathbb{H}$ is a π_0 isomorphism, any extension of $\tau: \mathrm{ko} \rightarrow \ell(\mathbb{R})$ along $\mathrm{ko} \rightarrow \mathrm{ksp}$ is also a π_0 isomorphism. Therefore, by Lemma 4.3, it suffices to show that the composite

$$\Sigma \mathrm{ko}/2 \xrightarrow{\tilde{\eta}} \mathrm{ko} \xrightarrow{\tau} \ell(\mathbb{R})$$

is ko -linearly null-homotopic. But we have

$$\mathrm{Map}_{\mathrm{ko}}(\Sigma \mathrm{ko}/2, \ell(\mathbb{R})) \simeq \mathrm{Map}_{\mathrm{ko}}(\Sigma \mathrm{ko}, \Omega \ell(\mathbb{R})/2) \simeq \Omega^{\infty+2} \ell(\mathbb{R})/2$$

which is connected. \square

We are now ready to prove Theorem B from the introduction, which we state here in a form better suited for describing the comparison map τ on homotopy groups in all non-negative degrees, see Proposition 5.1. We recall that $\mathrm{K}_*(A)[\eta]$ denotes the η -torsion of $\mathrm{K}_*(A)$, that is, the kernel of the map $\mathrm{K}_*(A) \rightarrow \mathrm{K}_{*+1}(A)$ given by multiplication by η . The cokernel of this map is denoted by $\mathrm{K}_{*+1}(A)/\eta$.

Theorem 4.5. *Let A be a C^* -algebra. For all $n \in \mathbb{Z}$, there are canonical and natural isomorphisms*

- (1) $\mathrm{L}_{4n}(A) \cong \mathrm{K}_{8n}(A)$,
- (2) $\mathrm{L}_{4n+1}(A) \cong \mathrm{K}_{8n+1}(A)/\eta$,
- (3) $\mathrm{L}_{4n+2}(A) \cong \mathrm{K}_{8n+6}(A)[\eta]$, and
- (4) $\mathrm{L}_{4n+3}(A) \cong \mathrm{K}_{8n+7}(A)$.

Proof. First, we note that it suffices to prove the theorem for $n = 0$, as the L-groups are naturally 4-periodic and the K-groups are naturally 8-periodic. Moreover, we note that the periodicity generators $b \in L_4(\mathbb{R})$ and $\beta_{\mathbb{R}} \in K_8(\mathbb{R})$ are canonical (not only up to sign), for instance because they are determined by squares in $L_4(\mathbb{C})$ and $K_8(\mathbb{C})$ respectively. Using the presentation $\ell(A) \simeq k(A) \otimes_{\text{ko}} \ell(\mathbb{R})$ obtained in Theorem A and Corollary 4.4 we deduce that the map $\text{ksp} \rightarrow \ell(\mathbb{R})$ induces the equivalence

$$\tau_{\leq 3}(k(A) \otimes_{\text{ko}} \text{ksp}) \xrightarrow{\cong} \tau_{\leq 3}\ell(A).$$

We now utilise that $k(A) = \tau_{\geq 0}K(A)$ is the connective cover of a KO-module and proceed with the following general observation. We let M be a KO-module and are then interested in the low degree homotopy of the ko-module

$$(\tau_{\geq 0}M) \otimes_{\text{ko}} \text{ksp}.$$

From the fibre sequence $\Sigma\text{ko}/2 \rightarrow \text{ko} \rightarrow \text{ksp}$ obtained in Lemma 4.3, we deduce that

- (1) $\pi_0(\tau_{\geq 0}M \otimes_{\text{ko}} \text{ksp}) \cong \pi_0(M)$, and
- (2) $\pi_1(\tau_{\geq 0}M \otimes_{\text{ko}} \text{ksp}) \cong \text{coker}(\pi_0(M) \xrightarrow{\eta} \pi_1(M))$.

To calculate π_2 and π_3 , we consider the following diagram of horizontal and vertical fibre sequences

$$\begin{array}{ccccc} \Sigma(\tau_{\geq 0}M)/2 & \longrightarrow & \tau_{\geq 0}M & \longrightarrow & \tau_{\geq 0}M \otimes_{\text{ko}} \text{ksp} \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma M/2 & \longrightarrow & M & \longrightarrow & M \otimes_{\text{KO}} \text{KSp} \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma(\tau_{< 0}M)/2 & \longrightarrow & \tau_{< 0}M & \longrightarrow & C \end{array}$$

from which we deduce that $\pi_i(C) = 0$ for $i \geq 3$ and that

$$\pi_2(C) \cong \pi_1(\Sigma(\tau_{< 0}M)/2) \cong \pi_{-1}(M)[2].$$

In addition, we note that $\text{KSp} \simeq \Sigma^4\text{KO}$. We therefore have a fibre sequence

$$\tau_{\geq 0}M \otimes_{\text{ko}} \text{ksp} \longrightarrow \Sigma^4M \longrightarrow C$$

whose long exact sequence on homotopy groups reveals that $\pi_3(\tau_{\geq 0}M \otimes_{\text{ko}} \text{ksp}) \cong \pi_3(\Sigma^4M) \cong \pi_{-1}M$ and that there is an exact sequence

$$0 \longrightarrow \pi_2(\tau_{\geq 0}M \otimes_{\text{ko}} \text{ksp}) \longrightarrow \pi_{-2}(M) \longrightarrow \pi_{-1}(M).$$

Here, we have used that $\pi_2C \subseteq \pi_{-1}(M)$. The latter map in this exact sequence is a natural transformation of functors $\pi_{-2} \rightarrow \pi_{-1}$ on KO-modules, and is therefore either trivial or the η -multiplication. We claim that it is the η -multiplication, which then shows the theorem.

The claim is equivalent to the statement that the map

$$\pi_2(\tau_{\geq 0}M \otimes_{\text{ko}} \text{ksp}) \longrightarrow \pi_{-2}(M)$$

appearing above is in general not an isomorphism. Therefore, it suffices to find an example of a KO-module M where $\pi_{-2}M \neq 0$ but $\pi_2(\tau_{\geq 0}M \otimes_{\text{ko}} \text{ksp}) = 0$. First, we note that there is an equivalence

$$\pi_2(\tau_{\geq 0}M \otimes_{\text{ko}} \text{ksp}) \simeq \pi_2(\tau_{[0,2]}M \otimes_{\text{ko}} \mathbb{Z})$$

Choosing $M = \text{KO}[-3]$, we find $\pi_{[0,2]}M = \mathbb{Z}[1, 3]$, so that

$$\pi_2(\tau_{[0,2]}M \otimes_{\text{ko}} \mathbb{Z}) \cong \pi_1(\mathbb{Z} \otimes_{\text{ko}} \mathbb{Z}) = 0.$$

However, $\pi_{-2}M = \pi_1\text{KO} \neq 0$, so the claim is shown. \square

We end this section with the following perspective on the map $k(A) \otimes_{\text{ko}} \text{ksp} \rightarrow \ell(A)$ which was used in the proof of Theorem 4.5. Namely, as a consequence of Theorem 4.1 and Lemma 4.3, there

is a commutative diagram

$$\begin{array}{ccccc}
\Sigma k(A)/2 & \longrightarrow & k(A) & \longrightarrow & k(A) \otimes_{k_0} \text{ksp} \\
\downarrow & & \parallel & & \downarrow \\
\widetilde{k(A)}_{hC_2} & \longrightarrow & k(A) & \longrightarrow & \ell(A)
\end{array}$$

and the fact that the right vertical map induces an equivalence of 3-truncations can be used to show that the cofibre of the left most vertical map is 3-connective with π_3 given by $k_0(A)/2$. Since this map is obtained from the map $\Sigma \mathbb{S}/2 \rightarrow \widetilde{\mathbb{S}}_{hC_2}$ upon applying the functor $- \otimes k(A)$, this result also follows from the following lemma.

Lemma 4.6. *There is a map $\Sigma \mathbb{S}/2 \rightarrow \Sigma^\infty BC_2$ whose cofibre is 3-connective with π_3 isomorphic to $\mathbb{Z}/2$.*

Proof. First, we recall the low dimensional homotopy groups of $\mathbb{S}/2$ and $\Sigma^\infty BC_2$: We have that $\pi_0(\mathbb{S}/2) = \mathbb{Z}/2$, $\pi_1(\mathbb{S}/2) = \mathbb{Z}/2$ and $\pi_2(\mathbb{S}/2) = \mathbb{Z}/4$. In addition, the η -multiplications

$$\pi_0(\mathbb{S}/2) \longrightarrow \pi_1(\mathbb{S}/2) \longrightarrow \pi_2(\mathbb{S}/2)$$

are injective, as follows from comparing with \mathbb{S} along the canonical map $\mathbb{S} \rightarrow \mathbb{S}/2$. Now, according to [Liu63], we have $\pi_1(\Sigma^\infty BC_2) = \mathbb{Z}/2$, $\pi_2(\Sigma^\infty BC_2) = \mathbb{Z}/2$ and $\pi_3(\Sigma^\infty BC_2) = \mathbb{Z}/8$. The Atiyah–Hirzebruch spectral sequence then shows that the map $\Sigma \mathbb{S} = \Sigma^\infty B\mathbb{Z} \rightarrow \Sigma^\infty BC_2$ induces the projection on π_1 an isomorphism on π_2 and an injection on π_3 . In particular, this map descends to a map $\Sigma \mathbb{S}/2 \rightarrow \Sigma^\infty BC_2$ and the induced map then induces an isomorphism on π_0 and π_1 . On π_3 , the composite $\Sigma \mathbb{S} \rightarrow \Sigma \mathbb{S}/2 \rightarrow \Sigma^\infty BC_2$ identifies with

$$\mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/8$$

where the composite is the non-trivial map. It follows that $\mathbb{Z}/4 \rightarrow \mathbb{Z}/8$ must be injective as claimed. This calculation also shows that the cofibre of the map $\Sigma \mathbb{S}/2 \rightarrow \Sigma^\infty BC_2$ has π_3 isomorphic to $\text{coker}(\mathbb{Z}/4 \subseteq \mathbb{Z}/8) \cong \mathbb{Z}/2$ as claimed. \square

5. ALGEBRAIC STRUCTURE OF $L_*(-)$

In this section we will describe the algebraic structure on the L-theory groups under the isomorphisms obtained in Theorem 4.5 and compare our results to previously known results. We will freely use the isomorphisms of Theorem 4.5 which identifies all L-groups.

Recall that the homotopy groups $KO_* = K_*(\mathbb{R})$ are 8-periodic with the (invertible) real Bott element $\beta_{\mathbb{R}}$ in degree 8. We fix the generator $x \in K_4(\mathbb{R}) \cong \mathbb{Z}$ whose complexification is $2\beta_{\mathbb{C}}^2$ and recall the relations $x^2 = 4\beta_{\mathbb{R}}$ and $\eta x = 0$.

Proposition 5.1. *For a C^* -algebra A , the map $\tau_A : k(A) \rightarrow \ell(A)$ induces the following maps on homotopy groups π_n for $n \geq 0$:*

$$\begin{aligned}
(2x)^n : K_{4n}(A) &\longrightarrow K_{8n}(A) \cong L_{4n}(A) \\
(2x)^n : K_{4n+1}(A) &\longrightarrow K_{8n+1}(A)/\eta \cong L_{4n+1}(A) \\
(2x)^n \cdot x : K_{4n+2}(A) &\longrightarrow K_{8n+6}(A)[\eta] \cong L_{4n+2}(A) \\
(2x)^n \cdot x : K_{4n+3}(A) &\longrightarrow K_{8n+7}(A) \cong L_{4n+3}(A) .
\end{aligned}$$

Remark 5.2. In particular, in degree $8n$ and $8n + 1$ the map under investigation is given by multiplication by 16^n , up to Bott periodicity isomorphisms. Since the map $K_4(\mathbb{R}) \rightarrow K_4(\mathbb{C})$ sends x to $2\beta_{\mathbb{C}}^2$, Proposition 5.1 also shows that for complex C^* -algebras, the map induces multiplication by 2^n on π_{2n} and π_{2n+1} . This was previously obtained in [LN18, Theorem 4.1] and we shall make use of this fact below.

Proof of Proposition 5.1. We note that the assignment $A \mapsto K_n(A)$ viewed as a functor $\text{KK} \rightarrow \text{Ab}$ is corepresented by an n -fold shift (i.e. suspension) of \mathbb{R} , which we denote by $\mathbb{R}[n]$. Therefore, the Yoneda lemma for product preserving functors $\text{KK} \rightarrow \text{Ab}$ implies that natural transformations $K_n \rightarrow L_n$ are in 1-1 correspondence to classes in $L_n(\mathbb{R}[n])$. By Theorem B, this group is isomorphic

to $K_0(\mathbb{R}) = \mathbb{Z}$ and $K_4(\mathbb{R}) = \mathbb{Z}\{x\}$ when $n \equiv 0, 1 \pmod{4}$ and $n \equiv 2, 3 \pmod{4}$, respectively. We deduce that maps $K_n(A) \rightarrow L_n(A)$ have to be given by multiplication with a multiple of x or a multiple of 1 (under the respective identifications depending on n described above). From the case of complex C^* -algebras as discussed in Remark 5.2, we immediately deduce the precise form of the multiple: we simply note that for a complex C^* -algebra A the element x acts as $2\beta_{\mathbb{C}}^2$. Therefore, $(2x)^n$ acts as 2^{2n} , and $(2x)^n \cdot x$ acts as 2^{2n+1} . Therefore, the formulas described in the statement of Proposition 5.1 are correct for complex C^* -algebras (this is the content of Remark 5.2), and hence by the above analysis in general. \square

Next, we want to explain how the lax symmetric monoidal structure on $L_*(-)$ is described in terms of the lax symmetric monoidal structure on $K_*(-)$ under the isomorphisms provided by Theorem B. To state the result we have to describe the maps

$$L_i(A) \otimes L_j(B) \longrightarrow L_{i+j}(A \otimes B)$$

for $i, j = 0, 1, 2, 3 \pmod{4}$ as everything is multiplicatively 4-periodic. By graded symmetry, it is enough to do this for $0 \leq i \leq j \leq 3$. We denote the lax symmetric monoidal structure of K-theory as

$$K_i(A) \otimes K_j(B) \longrightarrow K_{i+j}(A \otimes B) \quad (a, b) \mapsto a * b$$

and the induced $KO_* = K_*(\mathbb{R})$ -module structure on $K_*(A)$ by the multiplication sign.

Proposition 5.3. *Under the isomorphisms of Theorem 4.5 the exterior multiplication maps on the L-groups are maps of the following kind.*

- (1) $K_{8n}(A) \otimes K_{8m}(B) \longrightarrow K_{8n+8m}(A \otimes B)$
- (2) $K_{8n}(A) \otimes K_{8m+1}(B)/\eta \longrightarrow K_{8n+8m+1}(A \otimes B)/\eta$
- (3) $K_{8n}(A) \otimes K_{8m+6}(B)[\eta] \longrightarrow K_{8n+8m+6}(A \otimes B)[\eta]$
- (4) $K_{8n}(A) \otimes K_{8m+7}(B) \longrightarrow K_{8n+8m+7}(A \otimes B)$
- (5) $K_{8n+1}(A)/\eta \otimes K_{8m+1}(B)/\eta \longrightarrow K_{8n+8m+6}(A \otimes B)[\eta]$
- (6) $K_{8n+1}(A)/\eta \otimes K_{8m+6}(B)[\eta] \longrightarrow K_{8n+8m+7}(A \otimes B)$
- (7) $K_{8n+1}(A)/\eta \otimes K_{8m+7}(B) \longrightarrow K_{8n+8m+8}(A \otimes B)$
- (8) $K_{8n+6}(A)[\eta] \otimes K_{8m+6}(A)[\eta] \longrightarrow K_{8n+8m+8}(A \otimes B)$
- (9) $K_{8n+6}(A)[\eta] \otimes K_{8m+7}(A) \longrightarrow K_{8n+8m+9}(A \otimes B)/\eta$
- (10) $K_{8n+7}(A) \otimes K_{8m+7}(A) \longrightarrow K_{8n+8m+14}(A \otimes B)[\eta]$

For a belonging to the left tensor factor and b belonging to the right tensor factor, these maps are given by the following formulas:

	$K_{8m}(B)$	$K_{8m+1}(B)/\eta$	$K_{8m+6}(B)[\eta]$	$K_{8m+7}(B)$
$K_{8n}(A)$	$a * b$	$a * b$	$a * b$	$a * b$
$K_{8n+1}(A)/\eta$		$x \cdot (a * b)$	$a * b$	$2(a * b)$
$K_{8n+6}(A)[\eta]$			$\frac{x}{2\beta_{\mathbb{R}}}(a * b)$	$\frac{x}{2\beta_{\mathbb{R}}}(a * b)$
$K_{8n+7}(A)$				$2(a * b)$

Here, the abusive term $\frac{x}{2\beta_{\mathbb{R}}}(a * b)$ denotes an element depending naturally on a and b and whose multiplication with 2 is given by $\frac{x}{\beta_{\mathbb{R}}}(a * b)$. Part of the statement is the claim that there is a unique such element.

Proof. We take a step back again and recall that the exterior multiplication maps on L-theory are natural transformations

$$L_i \otimes L_j \rightarrow L_{i+j}.$$

If i and j are 0 or 3 modulo 4, then the L-groups are isomorphic to K-groups and thus the source $L_i \otimes L_j$ is corepresentable by shifts of \mathbb{R} (on the category $h\text{KK} \otimes h\text{KK}$ whose objects are pairs of C^* -algebras and whose hom abelian groups are the tensor products of the hom abelian groups in $h\text{KK}$). Consequently, the exterior multiplication $L_i \otimes L_j \rightarrow L_{i+j}$ is given by an element in $L_{i+j}(\Sigma^m \mathbb{R})$ for appropriate m . This group is isomorphic to $K_0(\mathbb{R}) = \mathbb{Z}$ and $K_0(\mathbb{R})[\eta] = 2\mathbb{Z}$ (depending on the precise values of i and j) so that in these cases the multiplication has to be

given by a multiple of $a * b$ and $2(a * b)$, respectively. Using that the map of Proposition 5.1 is to be compatible with external products, we immediately get the desired multiples. This proves cases (1), (4), and (10).

By Theorem B and the remark following Theorem B in the introduction, we have natural surjections $K_1(A) \rightarrow L_1(A)$ and $K_0(A_{\mathbb{C}}) \rightarrow L_2(A)$. The functor $A \mapsto K_0(A_{\mathbb{C}})$ is corepresentable by \mathbb{C} and the functor $A \mapsto K_1(A)$ by $\Sigma\mathbb{R}$. We deduce that for any values of i and j we have a surjection $F_i(A) \otimes F_j(A) \twoheadrightarrow L_i(A) \otimes L_j(A)$ where F_i and F_j are corepresentable. Any natural transformation with source $L_i \otimes L_j$ is then uniquely determined by its restriction to $F_i \otimes F_j$. Computing natural transformations $F_i \otimes F_j \rightarrow L_{i+j}$ the resulting groups are given by

$$K_0(\mathbb{R}), K_0(\mathbb{C}), K_4(\mathbb{R}), K_0(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}).$$

Using again that the comparison map τ is compatible with external products and Proposition 5.1, we obtain cases (2), (3), (5), (6), and (7). It remains to treat case (8) and (9). We shall argue case (8) and leave the details of case (9) to the reader. We consider the following diagram of natural transformations

$$\begin{array}{ccccc} K_2(A_{\mathbb{C}}) \otimes K_2(B_{\mathbb{C}}) & \xrightarrow{u \otimes u} & K_2(A) \otimes K_2(B) & \longrightarrow & K_4(A \otimes B) \\ \downarrow x \otimes x & & \downarrow x \otimes x & & \downarrow 2x \\ K_6(A_{\mathbb{C}}) \otimes K_6(B_{\mathbb{C}}) & \xrightarrow{u \otimes u} & K_6(A)[\eta] \otimes K_6(B)[\eta] & \xrightarrow{m} & K_8(A \otimes B) \end{array}$$

the left diagram commutes because the forgetful map $u: K(A_{\mathbb{C}}) \rightarrow K(A)$ is KO-linear. The right diagram commutes because it is the vertical maps are induced by τ , see Proposition 5.1, and τ is compatible with exterior multiplications. The lower right horizontal map m is the one we wish to describe as $m(a, b) = \frac{x}{2\beta_{\mathbb{R}}}(a * b)$. Since the lower left horizontal arrow is surjective, it suffices to show that the equality holds after precomposing with this surjective map. Doing this, both terms are natural transformations which, by corepresentability of the source, are given by elements of $K_0(\mathbb{C} \otimes \mathbb{C})$. Since this group is torsion free, we may equivalently show that

$$16m(u(a), u(b)) = \frac{8x}{\beta_{\mathbb{R}}}(u(a) * u(b))$$

where $a \in K_6(A_{\mathbb{C}})$ and $b \in K_6(B_{\mathbb{C}})$. Using the above commutative diagram, and the fact that the on the K-theory of complex C^* -algebras, x acts via $2\beta_{\mathbb{C}}^2$, we see that

$$4m(u(a), u(b)) = 2x \cdot (u(\beta_{\mathbb{C}}^{-2}a) * u(\beta_{\mathbb{C}}^{-2}b))$$

so it suffices to show that

$$4\beta_{\mathbb{R}} \cdot (u(\beta_{\mathbb{C}}^{-2}a) * u(\beta_{\mathbb{C}}^{-2}b)) = 4(u(a) * u(b)).$$

This follows from the facts that $4\beta_{\mathbb{R}} = x^2$, $-*$ is KO-bilinear, u is KO-linear, and that $xa = 2\beta_{\mathbb{C}}^2a$ as already used earlier. Case (9) can be shown by a similar argument. \square

Remark 5.4. In this remark, we collect what was previously known about the L-groups of C^* -algebras.

- (1) There is a canonical signature-type isomorphism $L_0(A) \rightarrow K_0(A)$, see e.g. [Ros05, Theorem 1.6].
- (2) There are canonical isomorphisms $K_n(A)[\frac{1}{2}] \cong L_n(A)[\frac{1}{2}]$, see [Ros05, Theorem 1.11] and [LN18] for the more general statement that there is a canonical equivalence $K[\frac{1}{2}] \simeq L[\frac{1}{2}]$ of spectrum valued functors.
- (3) For a unital real C^* -algebra A , there is a canonical surjection $K_1(A) \rightarrow L_1^h(A)$ whose kernel is generated by the image of $K_1(\mathbb{R}) \rightarrow K_1(A)$, see [Ros05, Theorem 1.9]. Here, L^h refers to *free* L-theory.

To the best of our knowledge, no conjectural relation between $L_n(A)$ and $K_n(A)$ has been made for $n \geq 2$ without inverting 2. We also note that, by construction, the isomorphism in (1) is the inverse of the canonical isomorphism induced by the map τ of Theorem A.

We will now also comment how our results imply statements about the higher free L-groups of unital C^* -algebras and in particular reprove part (3) above in Corollary 5.8. First, we describe the free L-theory of a unital C^* -algebra as follows. We recall that $SA = C_0((0, 1); A)$ denotes the C^* -algebraic suspension of the algebra A and note that S descends to the loop functor on the stable ∞ -category KK .

Proposition 5.5. *Let A be a unital C^* -algebra. There is a canonical fibre sequence*

$$\Sigma L(SA) \longrightarrow L^h(A) \longrightarrow C^{tC_2}$$

where $C = \ker(K_0(A) \rightarrow \tilde{K}_0(A)) = \text{Im}(K_0(\mathbb{R}) \rightarrow K_0(A))$ is a cyclic group.

Proof. By [LN18, Proposition 4.6] and the Rothenberg sequence for $L^h(-)$ and $L(-)$ [Ran80, §9], we have a commutative diagram of fibre sequences

$$(5.1) \quad \begin{array}{ccccc} \Sigma L(SA) & \longrightarrow & L(A) & \longrightarrow & K_0(A)^{tC_2} \\ \downarrow & & \parallel & & \downarrow \\ L^h(A) & \longrightarrow & L(A) & \longrightarrow & \tilde{K}_0(A)^{tC_2} \end{array}$$

from which the proposition follows immediately. \square

The following is an amusing consequence.

Corollary 5.6. *Suppose A is a unital C^* -algebra in which the element $[A] \in K_0(A)$ has odd order³. Then the map $\Sigma L(SA) \rightarrow L^h(A)$ is an equivalence.*

Proof. The element $[A] \in K_0(A)$ generates the kernel of the map $K_0(A) \rightarrow \tilde{K}_0(A)$. Therefore, under the assumptions of the corollary, C is a finite group of odd order, so its C_2 -Tate cohomology vanishes. \square

We then investigate the long exact sequence associated to the fibre sequence of Proposition 5.5. To do so, we first analyse the top horizontal fibre sequence in diagram (5.1) and recall that, since the C_2 -action on $K_0(A)$ is trivial⁴, we have $\hat{H}^{\text{ev}}(C_2; K_0(A)) \cong K_0(A)/2$ and $\hat{H}^{\text{odd}}(C_2; K_0(A)) \cong K_0(A)[2]$.

Proposition 5.7. *Under the isomorphisms provided by Theorem B, the natural maps $L_n(A) \rightarrow \pi_n(K_0(A)^{tC_2})$ appearing in the long exact sequence associated to the top horizontal fibre sequence of diagram (5.1) are the following ones:*

- (1) the projection $K_0(A) \rightarrow K_0(A)/2$ for $n \equiv 0 \pmod{4}$,
- (2) the trivial map $K_1(A)/\eta \rightarrow K_0(A)[2]$ for $n \equiv 1 \pmod{4}$,
- (3) the unique, non-trivial natural map $K_6(A)[\eta] \rightarrow K_0(A)/2$ for $n \equiv 2 \pmod{4}$ ⁵, and
- (4) the multiplication by η map $K_7(A) \rightarrow K_0(A)[2]$ for $n \equiv 3 \pmod{4}$.

Proof. First, we note that all maps appearing are natural in A . Next, we recall that the map under consideration is the composite of the natural transformations $L \rightarrow K_{\text{alg}}(-)^{tC_2} \rightarrow K_0(-)^{tC_2}$, both of which are canonically lax symmetric monoidal transformations. We deduce that the map under consideration is 4-periodic (since everything is a module over $L(\mathbb{R})$), hence it suffices to treat the cases $n = 0, 1, 2, 3$. The case $n = 0$ follows from a direct inspection. The case $n = 1$ is obtained by considering the natural maps

$$K_1(A) \longrightarrow K_1(A)/\eta \longrightarrow K_0(A)[2] \subseteq K_0(A)$$

³e.g. $A = \mathcal{O}_{2n}^{\mathbb{R}}$.

⁴any finitely generated projective A -module P admits a positive definite unimodular form, giving an isomorphism from P to P^{\vee} . See also the proof of Theorem 4.1 for the triviality of the C_2 -action on the spectrum $\text{k}(A)$.

⁵The assertion is that there is exactly one such natural map. An explicit description can be given as follows: lift an element in $K_6(A)[\eta]$ to an element in $K_6(A_{\mathbb{C}})$ along the ‘forgetful’ map $K_*(A_{\mathbb{C}}) \rightarrow K_*(A)$. Then multiply the lift with β^{-3} to obtain an element in $K_0(A_{\mathbb{C}})$ and apply the forgetful map followed by the mod 2 reduction.

and observing that any such natural map is given by multiplication by an element of $K_{-1}(\mathbb{R}) = 0$. Since the first map above is surjective and the last map is injective, the middle map is trivial as claimed. The case $n = 2$ is obtained by noting that the composite

$$K_6(A_{\mathbb{C}}) \longrightarrow K_6(A)[\eta] \longrightarrow K_0(A)/2$$

is again natural and the first map is surjective by the generalised Wood sequence discussed in the remark following Theorem B. Furthermore, as before, the source is, as a functor in A , corepresented by \mathbb{C} . Therefore, natural such maps are given by an element in $K_0(\mathbb{C})/2 = \mathbb{Z}/2$. It then suffices to show that the map under investigation is not trivial. This follows from the case of complex C^* -algebras: The 2-periodicity of L-theory for complex C^* -algebras indeed shows that this map identifies with the map for $n = 0$ which is non-trivial by the first part. This construction also shows that the description given in footnote 5 is correct: the map $K_6(A_{\mathbb{C}}) \rightarrow K_0(A)/2$ given by the non-trivial element $K_0(\mathbb{C})/2 = \mathbb{Z}/2$ factors as $K_6(A_{\mathbb{C}}) \rightarrow K_0(A_{\mathbb{C}}) \rightarrow K_0(A) \rightarrow K_0(A)/2$ since the element in $K_0(\mathbb{C})/2$ lifts through the induced maps $K_0(\mathbb{C}_{\mathbb{C}}) \rightarrow K_0(\mathbb{C}) \rightarrow K_0(\mathbb{C})/2$ in the prescribed manner.

Finally, the case $n = 3$ must, by the same reasoning as earlier, be given by multiplication with a 2-torsion element of $K_1(\mathbb{R}) = \mathbb{Z}/2\{\eta\}$. It then suffices to know that this map is non-trivial, which follows from considering the algebra $A = S\mathbb{R}$. \square

Corollary 5.8. *Let A be a unital C^* -algebra. Then*

- (1) *there is a canonical isomorphism $L_1^h(A) \cong K_1(A)/\langle \eta \rangle$, and*
- (2) *there is a canonical isomorphism $L_3^h(A) \cong K_7(A) \times_{K_8(A)} C$, where the maps appearing in the pullback are given by the η multiplication $K_7(A) \rightarrow K_8(A)$ and the canonical map $C \rightarrow K_0(A) \cong K_8(A)$.*

Proof. To prove part (1), we consider the following diagram

$$\begin{array}{ccccccc} C/2 & \longrightarrow & L_0(SA) & \longrightarrow & L_1^h(A) & \xrightarrow{0} & C[2] \\ & & \downarrow & & \parallel & & \downarrow \\ L_2(A) & \longrightarrow & K_0(A)/2 & \xrightarrow{\eta} & L_0(SA) & \longrightarrow & L_1(A) \longrightarrow K_0(A)[2] \hookrightarrow L_{-1}(SA) \end{array}$$

and note first that $L_0(SA) \cong K_1(A)$ by Theorem B. Then we observe that the right most vertical arrow is injective, simply because $C \rightarrow K_0(A)$ is. Furthermore, the right most bottom horizontal arrow is injective, see [Ros05, Remark 1.10] and the argument used in the proof of [LN18, Proposition 4.6] via diagram (1) therein. It follows that the right most top horizontal arrow is trivial, so that $L_1^h(A)$ is a natural quotient of $K_1(A)$. To see which precise quotient it is, we observe again by naturality, that the second to left most bottom horizontal arrow is given by multiplication by η : It is either that or trivial, and the case of $A = \mathbb{R}$ shows that the map is non-trivial since $L_2(\mathbb{R}) = 0$.

To prove part (2), we consider the same exact sequences shifted in the appropriate degrees:

$$\begin{array}{ccccccc} L_2(SA) & \longrightarrow & L_3^h(A) & \longrightarrow & C[2] \subseteq C \\ & & \parallel & & \downarrow & & \downarrow \\ L_4(A) & \twoheadrightarrow & K_0(A)/2 & \xrightarrow{0} & L_2(SA) & \hookrightarrow & L_3(A) \longrightarrow K_0(A)[2] \subseteq K_0(A) \end{array}$$

where the left most bottom arrow is surjective by Proposition 5.7. Consequently, the map $L_3^h(A) \rightarrow L_3(A)$ is injective. The claim then follows from the isomorphism $L_3(A) \cong K_7(A)$ of Theorem 4.5 and the fact established in Proposition 5.7 under this isomorphism, the map $L_3(A) \rightarrow K_0(A)$ appearing in the above diagram as the right most bottom horizontal map is given by multiplication by η . \square

Finally, we say as much as we can about $L_2^h(A)$:

Proposition 5.9. *Let A be a unital C^* -algebra. Then there is an exact sequence*

$$C[2] \longrightarrow K_2(A)/\eta \xrightarrow{\tilde{x}} L_2^h(A) \longrightarrow C/2 \xrightarrow{\eta} K_1(A)$$

where \tilde{x} is a map whose composition with the canonical map $L_2^h(A) \rightarrow L_2(A) \cong K_6(A)[\eta]$ is given by multiplication by x .

Proof. We inspect the long exact sequence associated to the fibre sequence of Proposition 5.5 and use that $L_1(SA) \cong K_2(A)/\eta$ and $L_0(SA) \cong K_1(A)$ by Theorem 4.5. To see the claim about the composite of \tilde{x} with the map $L_2^h(A) \rightarrow L_2(A) \cong K_6(A)[\eta]$, we note that again by naturality, this composite is given by a multiple of the x multiplication. The case $A = \mathbb{C}$ then shows the claim. \square

Remark 5.10. The sequence of Proposition 5.9 can of course simplify: For instance, if C has odd order, or when $C[2] = 0$ and $0 \neq \eta \in K_1(A)$, we find that $L_2^h(A) \cong K_2(A)/\eta$.

Remark 5.11. The map $C[2] \rightarrow K_2(A)/\eta$ appearing in the sequence of Proposition 5.9 picks out a particular element of the target (recall that $C[2]$ is either cyclic of order 2 or trivial). Under the isomorphism

$$K_2(A)/\eta \cong \ker(K_0(A_{\mathbb{C}}) \rightarrow K_0(A))$$

induced by the Wood sequence, this element is given by the composite

$$C[2] \longrightarrow K_0(A)[2] \longrightarrow \ker(K_0(A_{\mathbb{C}}) \rightarrow K_0(A))$$

where we claim that the latter map is induced the canonical map $K_0(A) \rightarrow K_0(A_{\mathbb{C}})$ (which, when restricted to 2-torsion lands in the indicated kernel since the composite $K_0(A) \rightarrow K_0(A_{\mathbb{C}}) \rightarrow K_0(A)$ is given by multiplication by 2). Indeed, this map induces a natural map

$$K_1(A/2) \longrightarrow K_0(A)[2] \longrightarrow \ker(K_0(A_{\mathbb{C}}) \rightarrow K_0(A)) \subseteq K_0(A_{\mathbb{C}})$$

which in turn determines the map in question, since the first map is surjective. This composite is determined by an element of $K_0(\mathbb{C}/2) \cong \mathbb{Z}/2$, since the source is corepresented by $\mathbb{R}/2$. It then suffices to note that the map in question and the proposed map are both natural and non-trivial. To see that the map $C[2] \rightarrow K_2(A)/\eta$ appearing in Proposition 5.9 is non-trivial, we can consider the case $A = \mathcal{O}_3^{\mathbb{C}}$. It satisfies $K_0(A) = \mathbb{Z}/2$ and $\tilde{K}_0(A) = 0$. In particular $C = \mathbb{Z}/2$ and the map $L^h(A) \rightarrow L(A)$ is an equivalence. Since $K_1(A) = 0$ we deduce that $L_1(A) \cong L_3^h(A) = 0$. This shows that the map $C[2] \rightarrow K_2(A)/\eta$ is injective (and therefore in fact bijective). The same example also shows that the map $K_0(A)[2] = K_0(A) \rightarrow K_0(A_{\mathbb{C}})$ is non-trivial: Indeed, since A is complex there is an isomorphism $A_{\mathbb{C}} \cong A \times A$ under which the map from A corresponds to the diagonal.

Remark 5.12. Finally, we explain that for a unital C^* -algebra A , the map $\tau: k(A) \rightarrow L(A)$, on positive homotopy groups factors canonically through the canonical map $L^h(A) \rightarrow L(A)$. Indeed, we shall argue that there is a canonical map $k^{\text{free}}(A) \rightarrow L^h(A)$, where $k^{\text{free}}(A)$ denotes the group completion of the topological category of free A -modules, participating in the following commutative diagram.

$$\begin{array}{ccc} k^{\text{free}}(A) & \longrightarrow & L^h(A) \\ \downarrow & & \downarrow \\ k(A) & \longrightarrow & L(A) \end{array}$$

The left vertical map is induced by the canonical inclusion of free into projective modules and induces an equivalence on connected covers. Indeed, there is a commutative diagram

$$\begin{array}{ccccc} k^{\text{free}}(A) & \longrightarrow & \text{GW}_{\text{top}}^{\text{free}}(A) & \longrightarrow & L^h(A) \\ \downarrow & & \downarrow & & \downarrow \\ k(A) & \longrightarrow & \text{GW}_{\text{top}}(A) & \longrightarrow & L(A) \end{array}$$

where the maps from K-theory to Grothendieck–Witt theory equip a module over A with its unique positive definite form as described earlier.

Under the isomorphisms of Corollary 5.8, the map $k^{\text{free}} \rightarrow L^h$ on low homotopy groups is then given as follows:

- (1) the canonical projection $K_1(A) \rightarrow K_1(A)/\langle \eta \rangle \cong L_1^h(A)$,

- (2) the map $K_2(A) \rightarrow K_2(A)/\eta \xrightarrow{\tilde{x}} L_2^h(A)$ where \tilde{x} is as in Proposition 5.9, and
- (3) the map $K_3(A) \rightarrow K_7(A) \times_{K_8(A)} C \cong L_3^h(A)$ given by the x -multiplication – note that $\eta x = 0$, so the x multiplication on K_3 indeed has image in the claimed subgroup of $K_7(A)$.

6. EXAMPLES

In this section, we present a number of examples where we calculate L-groups of C^* -algebras. We note here, that due to the fact that the (graded) cohomological dimension of $\pi_*(L(\mathbb{R}))$ is 1, an $L(\mathbb{R})$ -module spectrum is (non-canonically) determined by its homotopy groups. Below, we shall therefore concentrate on calculating L-groups, and sometimes construct in addition canonical fibre sequences describing the L-spectrum. To do so efficiently, we begin with the following lemma.

Lemma 6.1. *We have a canonical equivalence $\mathbb{Z} \otimes_{\text{ko}} L(\mathbb{R}) \cong L(\mathbb{C})/2$.*

Proof. There is a canonical fibre sequence $\Sigma^2 \text{ku} \xrightarrow{\beta} \text{ku} \rightarrow \mathbb{Z}$. Therefore, applying $- \otimes_{\text{ko}} L(\mathbb{R})$ and Theorem A, there is a canonical fibre sequence

$$\Sigma^2 L(\mathbb{C}) \xrightarrow{2b_{\mathbb{C}}} L(\mathbb{C}) \rightarrow \mathbb{Z} \otimes_{\text{ko}} L(\mathbb{R}),$$

where $b_{\mathbb{C}} \in L_2(\mathbb{C}) \cong \mathbb{Z}$ denotes the generator such that $\tau(\beta) = 2b_{\mathbb{C}}$, see [LN18, Lemma 4.9]. Since $b_{\mathbb{C}}$ is invertible in $L_*(\mathbb{C})$, the lemma follows. \square

Example 6.2. Let A be a complex C^* -algebra. Then

$$L(A) \simeq k(A) \otimes_{\text{ko}} L(\mathbb{R}) \simeq k(A) \otimes_{\text{ku}} \text{ku} \otimes_{\text{ko}} L(\mathbb{R}) = k(A) \otimes_{\text{ku}} L(\mathbb{C}).$$

Furthermore, we recover that $L_0(A) \cong K_0(A)$ and $L_1(A) \cong K_1(A)$, since η is trivial on ku -modules.

Example 6.3. We have $\Sigma^n L(\mathbb{R}) \xrightarrow{\cong} L(\mathbb{R}[n])$ for $0 \leq n \leq 3$. Here, the notation $A[n]$ refers to the n -fold suspension of A in the stable ∞ -category KK ; we note that this construction is implemented by an appropriate C^* -algebraic suspension, that is, $A[-1]$ is represented by SA . Indeed, the example follows from the fibre sequence

$$\Sigma L(A[-1]) \rightarrow L(A) \rightarrow K_0(A)^{tC_2}$$

obtained in [LN18, Proposition 4.6] and the fact that $K_n(\mathbb{R}) = 0$ for $n = 5, 6, 7$.

Example 6.4. Let \mathbb{H} be the quaternions. Then we have $k(\mathbb{H}) = \text{ksp}$, and consequently, $\ell(\mathbb{H}) = \text{ksp} \otimes_{\text{ko}} \ell(\mathbb{R})$. Furthermore, the L-groups are given by

- (1) $L_0(\mathbb{H}) = K_0(\mathbb{H}) \cong \mathbb{Z}$,
- (2) $L_1(\mathbb{H}) = \text{coker}(K_0(\mathbb{H}) \rightarrow K_1(\mathbb{H})) = \text{coker}(\mathbb{Z} \rightarrow 0) = 0$,
- (3) $L_2(\mathbb{H}) = \ker(K_6(\mathbb{H}) \rightarrow K_7(\mathbb{H})) = \ker(\mathbb{Z}/2 \rightarrow 0) = \mathbb{Z}/2$, and
- (4) $L_3(\mathbb{H}) = K_7(\mathbb{H}) = 0$.

In addition, from the fibre sequence $\Sigma^4 \text{ko} \rightarrow \text{ksp} \rightarrow \mathbb{Z}$ and Lemma 6.1, we obtain a canonical fibre sequence

$$L(\mathbb{R}) \rightarrow L(\mathbb{H}) \rightarrow L(\mathbb{C})/2,$$

where the first map classifies 2 times a generator of $L_0(\mathbb{H})$.

Example 6.5. Since $\mathbb{H} = \mathbb{R}[4]$ in KK , we have already determined $L(\mathbb{R}[n])$ for $0 \leq n \leq 5$. In addition, similarly as in Example 6.3, we have that $\Sigma L(\mathbb{H}) \simeq L(\mathbb{H}[1])$ since $K_3(\mathbb{R}) = 0$. Since $\mathbb{R}[n] \simeq \mathbb{R}[n+8]$ in KK by real Bott periodicity, we shall now also calculate the L-groups of the remaining shifts of \mathbb{R} , namely $\mathbb{R}[6]$ and $\mathbb{R}[7]$. Here, we find

- (1) $L_0(\mathbb{R}[6]) = \mathbb{Z}/2$ and $L_0(\mathbb{R}[7]) = \mathbb{Z}/2$,
- (2) $L_1(\mathbb{R}[6]) = 0$ and $L_1(\mathbb{R}[7]) = 0$,
- (3) $L_2(\mathbb{R}[6]) = \mathbb{Z}$ and $L_2(\mathbb{R}[7]) = 0$, and
- (4) $L_3(\mathbb{R}[6]) = \mathbb{Z}/2$ and $L_3(\mathbb{R}[7]) = \mathbb{Z}$.

Example 6.6. As spectra, there is an equivalence $L(\mathbb{C}[1]) \simeq \Sigma L(\mathbb{C})$. However, the canonical map $\Sigma L(\mathbb{C}) \rightarrow L(\mathbb{C}[1])$ is *not* an equivalence, as follows again from [LN18, Proposition 4.6]: its cofibre is given by \mathbb{Z}^{tC_2} . In other words, the canonical map identifies with the times 2 map on $\Sigma L(\mathbb{C})$.

Example 6.7. Consider the algebra $C(\mathbb{T})$ of continuous real valued functions on the circle. Then we have an equivalence in KK given by $C(\mathbb{T}) = \mathbb{R} \oplus \mathbb{R}[-1]$. From this, the fact that L-theory preserves products, and Example 6.5 we obtain the following L-groups.

- (1) $L_0(C(\mathbb{T})) = \mathbb{Z} \oplus \mathbb{Z}/2$,
- (2) $L_1(C(\mathbb{T})) = 0$,
- (3) $L_2(C(\mathbb{T})) = 0$,
- (4) $L_3(C(\mathbb{T})) = \mathbb{Z}$.

Example 6.8. Let G be a torsionfree group for which the Baum–Connes conjecture holds. Then we get

- (1) $L_0(C_r^*G) = \text{KO}_0(BG)$,
- (2) $L_3(C_r^*G) = \text{KO}_{-1}(BG)$,

By Anderson duality, this shows that one can recover the abelian group $\text{KO}^4(BG)$ from $L_*(C_r^*G)$. Indeed, there is a short exact sequence

$$0 \longrightarrow \text{Ext}_{\mathbb{Z}}^1(\text{KO}_{-1}(BG), \mathbb{Z}) \longrightarrow \text{KO}^4(BG) \longrightarrow \text{Hom}_{\mathbb{Z}}(\text{KO}_0(BG), \mathbb{Z}) \longrightarrow 0$$

which splits non-canonically; see e.g. [HLN21] for a review of Anderson duality and the fact that the Anderson dual of KO is given by $\Omega^4\text{KO}$.

Example 6.9. The Baum–Connes conjecture is known to be true for free groups \mathcal{F}_n of rank $n \geq 1$. In particular, we have an equivalence $\text{K}(C_r^*\mathcal{F}_n) \simeq \text{KO} \oplus \bigoplus_n \text{KO}[1]$, and therefore obtain

$$L(C_r^*\mathcal{F}_n) \simeq L(\mathbb{R}) \oplus \bigoplus_n L(\mathbb{R})[1].$$

Remark 6.10. We warn the reader that, contrary to the complex case, $C_r^*\mathbb{Z}$ is not isomorphic to $C(\mathbb{T})$, but rather to the algebra of C_2 -equivariant continuous functions $\mathbb{T} \rightarrow \mathbb{C}$, where \mathbb{T} and \mathbb{C} both carry the complex conjugation action.

Example 6.11. Let Σ_g be a surface of genus $g \geq 2$ and π its fundamental group. The Baum–Connes conjecture is known for surface groups, so we find that

$$\text{K}(C_r^*\pi) \simeq \Sigma_{g,+} \otimes \text{KO} \simeq \text{KO} \oplus \text{KO}[1]^{\oplus 2g} \oplus \text{KO}[2].$$

Consequently, we obtain

$$L(C_r^*\pi) \simeq L(\mathbb{R}) \oplus L(\mathbb{R}) \oplus L(\mathbb{R})[1]^{\oplus 2g} \oplus L(\mathbb{R})[2].$$

Example 6.12. Let W be a right angled Coxeter group associated to a flag complex Σ as studied e.g. in [KLL21]. Then [KLL21, Theorem 7.16] shows that $\text{K}(C_r^*W) \simeq \text{KO}^r$, where r is the number of simplices of Σ , including the empty simplex. Therefore, we find $L(C_r^*W) \simeq L(\mathbb{R})^r$.

Remark 6.13. In fact, [KLL21] shows that there are explicit maps $\alpha_i: \mathbb{R} \rightarrow \mathbb{R}W \subseteq C_r^*W$, for $i = 1, \dots, r$, such that the induced maps

$$\bigoplus_r \text{KO} \longrightarrow \text{K}(C_r^*W) \text{ and } \bigoplus_r L(\mathbb{R}) \longrightarrow L(\mathbb{R}W)$$

are equivalences. Combined with Theorem A, we deduce that among the following two maps

$$\bigoplus_r L(\mathbb{R}) \longrightarrow L(\mathbb{R}W) \longrightarrow L(C_r^*W)$$

both the first map and the composite are equivalences. Therefore, so is the second map. This gives a non-trivial example where the completion conjecture of [LN18] is true *before* inverting 2.

Example 6.14. Let A be a real C^* -algebra equipped with an automorphism $\varphi: A \rightarrow A$. Then there is a fibre sequence

$$A \xrightarrow{\text{id}-\varphi} A \longrightarrow A \rtimes_{\varphi} \mathbb{Z}$$

in KK. If $\text{id} - \varphi_*$ is the zero map on $\text{K}_{-1}(A)$, one also obtains a fibre sequence

$$\text{k}(A) \xrightarrow{1-\varphi_*} \text{k}(A) \longrightarrow \text{k}(A \rtimes_{\varphi} \mathbb{Z})$$

of ko-modules. Consequently, there is then also a fibre sequence

$$L(A) \xrightarrow{1-\varphi} L(A) \longrightarrow L(A \rtimes_{\varphi} \mathbb{Z}).$$

More generally, there is a similar (conditional) fibre sequence describing the L-theory of reduced crossed products by free groups.

Example 6.15. An example of a crossed product by \mathbb{Z} is the real rotation algebra $A_{\theta} = C(\mathbb{T}) \rtimes_{\theta} \mathbb{Z}$ where θ is a real number and acts on functions on the circle group \mathbb{T} by a rotation by θ . However, by homotopy invariance, in KK we have an equivalence $A_{\theta} \simeq C(\mathbb{T}) \oplus C(\mathbb{T})[1] \simeq \mathbb{R} \oplus \mathbb{R}[-1] \oplus \mathbb{R}[1] \oplus \mathbb{R}$. We therefore obtain

$$L(A_{\theta}) \simeq L(\mathbb{R})^{\oplus 2} \oplus L(\mathbb{R}[-1]) \oplus L(\mathbb{R}[1]).$$

Using our previous calculations, we finally obtain

- (1) $L_0(A_{\theta}) = \mathbb{Z}^2 \oplus \mathbb{Z}/2$,
- (2) $L_1(A_{\theta}) = \mathbb{Z}$,
- (3) $L_2(A_{\theta}) = 0$, and
- (4) $L_3(A_{\theta}) = \mathbb{Z}$.

Example 6.16. We consider the real Cuntz algebras $\mathcal{O}_{n+1}^{\mathbb{R}}$. In KK, there is a canonical equivalence $\mathcal{O}_{n+1}^{\mathbb{R}} \simeq \mathbb{R}/n$. In particular, $k(\mathcal{O}_{n+1}^{\mathbb{R}}) \simeq ko/n$. Therefore, we find

$$L(\mathcal{O}_{n+1}^{\mathbb{R}}) \simeq ko/n \otimes_{ko} L(\mathbb{R}) = L(\mathbb{R})/n.$$

Likewise, one can consider the tensor products $\mathcal{O}_{n+1}^{\mathbb{R}} \otimes_{\mathbb{R}} \mathcal{O}_{m+1}^{\mathbb{R}}$, where one finds

$$L(\mathcal{O}_{n+1}^{\mathbb{R}} \otimes_{\mathbb{R}} \mathcal{O}_{m+1}^{\mathbb{R}}) \simeq L(\mathbb{R})/\gcd(m, n)$$

contrary to the case of K-theory, where the real K-groups of $\mathcal{O}_{n+1}^{\mathbb{R}} \otimes_{\mathbb{R}} \mathcal{O}_{m+1}^{\mathbb{R}}$ do not only depend on the greatest common divisor of m and n – simply because $(ko/n)/m$ does not only depend on this number; see [Boe02] for explicit calculations.

Therefore, not surprisingly, L-theory of *real* C^* -algebras is a strictly weaker invariant than K-theory: The real C^* -algebras $\mathcal{O}_3^{\mathbb{R}} \otimes_{\mathbb{R}} \mathcal{O}_5^{\mathbb{R}}$ and $\mathcal{O}_3^{\mathbb{R}} \otimes_{\mathbb{R}} \mathcal{O}_3^{\mathbb{R}}$ are distinguished by their K-groups, but not by their L-groups.

Example 6.17. Let $\mathcal{E}_{2n}^{\mathbb{R}}$ denote the simple separable nuclear real form of the Cuntz-algebra $\mathcal{O}_{2n+1}^{\mathbb{C}}$ considered in [BRS11]. Its topological K-theory is given by

$$K(\mathcal{E}_{2n}^{\mathbb{R}}) \simeq KO/nx$$

where $x \in \pi_4(KO)$ is a generator. Note that its complexification is given by $KU/2n\beta^2 \simeq KU/2n$, compatible with the equivalence $K(\mathcal{O}_{2n+1}^{\mathbb{C}}) \simeq KU/2n$. There is a fibre sequence

$$\mathbb{Z} \longrightarrow ko/nx \longrightarrow k(\mathcal{E}_{2n}^{\mathbb{R}})$$

where the first map induces multiplication by $4n$ on π_0 . We conclude that there is a fibre sequence

$$L(\mathbb{C})/2 \longrightarrow L(\mathbb{R})/8n \longrightarrow L(\mathcal{E}_{2n}^{\mathbb{R}})$$

in which the first map induces the non-zero map on π_{4i} . In particular, the L-groups are given by

- (1) $L_0(\mathcal{E}_{2n}^{\mathbb{R}}) = \mathbb{Z}/4n$,
- (2) $L_1(\mathcal{E}_{2n}^{\mathbb{R}}) = 0$,
- (3) $L_2(\mathcal{E}_{2n}^{\mathbb{R}}) = \mathbb{Z}/2$,
- (4) $L_3(\mathcal{E}_{2n}^{\mathbb{R}}) = \mathbb{Z}/2$.

We end with a number of structural examples:

Example 6.18. The L-groups of a C^* -algebra with $K_1(A_{\mathbb{C}}) = K_7(A) = K_6(A) = 0$, are concentrated in degrees divisible by 4.

Example 6.19. If the K-groups of a C^* -algebra are finitely generated, then so are the L-groups.

Example 6.20. If the K-groups of a C^* -algebra vanish after inverting a number n (e.g. when they are n -primary torsion), then so do the L-groups.

Example 6.21. Viewed as a functor on the bootstrap class \mathcal{B} , i.e. the thick subcategory of KK generated by \mathbb{R} , and as taking values in $L(\mathbb{R})$ -modules, L-theory is symmetric monoidal. More precisely, the lax symmetric monoidal structure on L-theory is in fact a symmetric monoidal structure. Indeed, the functor $L: \mathcal{B} \rightarrow \text{Mod}(L(\mathbb{R}))$ is given by the composite of the K-theory functor with the extension of scalars map $\text{Mod}(\text{KO}) \rightarrow \text{Mod}(L(\mathbb{R}))$ which is canonically symmetric monoidal. Moreover, the K-theory functor, when restricted to the bootstrap class is fully faithful (because \mathbb{R} is a generator of the bootstrap class by definition) and symmetric monoidal (because its left adjoint is a priori symmetric monoidal and therefore, when restricted to $\text{Perf}(\text{KO})$ is a symmetric monoidal equivalence).

7. INTEGRAL BAUM–CONNES AND FARRELL–JONES COMPARISON

In [LN18], we have used the equivalence $K[\frac{1}{2}] \simeq L[\frac{1}{2}]$ to compare the Baum–Connes assembly map and the L-theoretic Farrell–Jones assembly maps after inverting 2. The purpose of this section is to prove the following integral refinement of this result.

Theorem 7.1. *The map $\tau: k \rightarrow L$ induces the commutative diagram*

$$\begin{array}{ccc} \text{ko}_*^G(\underline{E}G) & \xrightarrow{\text{BC}} & k_*(C_r^*G) \\ \downarrow \tau & & \downarrow \tau \\ L\mathbb{R}_*^G(\underline{E}G) & \xrightarrow{\text{FJ}} & L_*(\mathbb{R}G) \longrightarrow L_*(C_r^*G) \end{array}$$

In order to explain the terms in the theorem, we will first briefly recall the setup for assembly maps as proposed by Davis and Lück [DL98].

The Davis–Lück picture for assembly maps starts with a discrete group G and an equivariant homology theory E , encoded as a functor $\text{Orb}(G) \rightarrow \text{Sp}$, where $\text{Orb}(G)$ is the *orbit category* of G , that is, the full subcategory of G -sets consisting of transitive G -sets (i.e. G -sets isomorphic to G/H for a subgroup H of G). A family \mathcal{F} of subgroups of G is a collection of subgroups closed under conjugation and passing to further subgroups. Associated to any such family \mathcal{F} , we may form the \mathcal{F} -orbit category $\text{Orb}_{\mathcal{F}}(G)$ which is the full subcategory on all transitive G -sets whose stabilisers belong to \mathcal{F} (i.e. G -sets isomorphic to G/H for $H \in \mathcal{F}$). The \mathcal{F} -assembly map for G and E is then given by the canonical map

$$E^G(E_{\mathcal{F}}G) \stackrel{\text{def}}{=} \text{colim}_{G/H \in \text{Orb}_{\mathcal{F}}G} E(G/H) \longrightarrow E(G/G).$$

Given an inclusion of families $\mathcal{F} \subseteq \mathcal{F}'$, there is an evident factorisation of the \mathcal{F} -assembly map as follows:

$$E^G(E_{\mathcal{F}}G) \longrightarrow E^G(E_{\mathcal{F}'}G) \longrightarrow E(G/G)$$

in which the first map is referred to as the *relative assembly map* (with respect to the inclusion of families $\mathcal{F} \subseteq \mathcal{F}'$) and the second map is the \mathcal{F}' -assembly map. Following standard notation we shall also write $E_{\mathcal{F}\text{in}}G = \underline{E}G$ and $E_{\mathcal{V}_{\text{cyc}}}G = \underline{\underline{E}}G$.

Relevant for us will be the functors given by equivariant topological K-theory and equivariant L-theory. These are functors

$$K^G, LR^G: \text{Orb}(G) \longrightarrow \text{Sp} \quad , \quad G/H \mapsto K(C^*H), L(RH)$$

for an involutive ring R , see e.g. [LN18] for further details. For the family of finite subgroups $\mathcal{F}\text{in}$, we shall denote these assembly maps by

$$\text{BC}: K_*^G(\underline{E}G) \longrightarrow K_*(C^*G) \quad \text{and} \quad \text{FJ}: LR_*^G(\underline{E}G) \longrightarrow L(RG).$$

We also note that the map FJ factors as the composite

$$LR_*^G(\underline{E}G) \longrightarrow LR_*^G(\underline{\underline{E}}G) \longrightarrow L(RG),$$

whose second map we shall later also denote by FJ.

Remark 7.2. We warn the reader that the L-theoretic Farrell–Jones conjecture is more specifically about the assembly map for the family $\mathcal{V}\text{cyc}$ of virtually cyclic subgroups and for a related (but in general different) functor $G/H \mapsto \mathbb{L}^q(RH)$ where \mathbb{L}^q is the Karoubi-localisation of L^q in the sense of [CDH⁺22], also known as *universally decorated* L-theory, denoted by $L^{(-\infty),q}$ in the literature, where the superscript q refers to quadratic rather than symmetric L-theory. We show in Theorem 7.6 below, that for a regular ring R and a torsion free group G , the Farrell–Jones conjecture implies that the map denoted FJ above is also an isomorphism (in fact, for either of the two maps denoted FJ above, as the first map in the composite is an isomorphism under the assumptions made, see Proposition 7.7); to the best of our knowledge, this had not been observed so far.

Remark 7.3. The assembly map in topological K-theory described above is related to the Baum–Connes conjecture which was originally phrased in terms of equivariant Kasparov theory. First and foremost, this conjecture is more specifically about the composite

$$K_*^G(\underline{E}G) \xrightarrow{\text{BC}} K_*(C^*G) \longrightarrow K_*(C_r^*G)$$

where $C^*G \rightarrow C_r^*G$ is the canonical morphism. We note here that the association $G \mapsto C_r^*G$ is not functorial in group homomorphisms, as famously the reduced group C^* -algebra $C_r^*\mathcal{F}_n$ of a non-abelian free group is a simple algebra [Pow75]. Therefore, the Davis–Lück picture for the assembly map in topological K-theory uses the full group C^* -algebra instead.

Now, it was shown in [Kra21] (and later and with different methods in [BEL21a]) that the assembly map for G , the family $\mathcal{F}\text{in}$ of finite subgroups of G , and the functor K^G is isomorphic to the Baum–Connes assembly map, and in [Lan15] that for torsion-free groups, this assembly map has an interpretation as taking a Mishchenko–Fomenko index (this latter result was a folklore result known to the experts for a long time).

In addition, the assembly map in topological K-theory is often performed using the complex group C^* -algebra rather than the real one, but see [Sch04] for a relation between the two.

The natural map $k \rightarrow L$ as functors on the category KK induces a natural transformation $k^G \rightarrow L\mathbb{R}^G$ as functors on the orbit category, see [LN18] for the details. Consequently, we obtain the following theorem, which is an integral analog of [LN18, Theorem D].

Theorem 7.4. *The map $\tau: k \rightarrow L$ of Theorem 4.1 induces the following commutative diagram.*

$$\begin{array}{ccc} \text{ko}_*^G(\underline{E}G) & \xrightarrow{\text{BC}} & k_*(C^*G) \\ \downarrow \tau & & \downarrow \tau \\ L\mathbb{R}_*^G(\underline{E}G) & \xrightarrow{\text{FJ}} & L_*(\mathbb{R}G) \longrightarrow L_*(C^*G) \end{array}$$

Proof of Theorem 7.1. There is a canonical map $C^*G \rightarrow C_r^*G$ from the full to the reduced group C^* -algebra. Since τ is natural, we obtain the commutative diagram

$$\begin{array}{ccccc} \text{ko}_*^G(\underline{E}G) & \xrightarrow{\text{BC}} & k_*(C^*G) & \longrightarrow & k_*(C_r^*G) \\ \downarrow \tau & & \downarrow \tau & & \downarrow \tau \\ L\mathbb{R}_*^G(\underline{E}G) & \xrightarrow{\text{FJ}} & L_*(\mathbb{R}G) & \longrightarrow & L_*(C_r^*G) \end{array}$$

which is the content of Theorem 7.1. □

Remark 7.5. Upon inverting 2 and the Bott element $\beta_{\mathbb{R}}$, the diagram of Theorem 7.1 becomes equivalent to the diagram

$$\begin{array}{ccc} \text{KO}_*^G(\underline{E}G)[\frac{1}{2}] & \longrightarrow & \text{KO}_*(C_r^*G)[\frac{1}{2}] \\ \downarrow \cong & & \downarrow \cong \\ L\mathbb{R}_*^G(\underline{E}G)[\frac{1}{2}] & \longrightarrow & L_*(\mathbb{R}G)[\frac{1}{2}] \longrightarrow L_*(C_r^*G)[\frac{1}{2}] \end{array}$$

which is the one obtained earlier in [LN18, Theorem D]. This uses in particular that the comparison map $\mathrm{LR}_*^G(\underline{EG}) \rightarrow \mathrm{LR}_*^G(\underline{EG})$ is an isomorphism after inverting 2 [LR05, Proposition 2.18]. Theorem 7.1 in addition provides some finer information about the comparison, as for instance the kernels and cokernels of the vertical maps appearing in the diagram of Theorem 7.1 can be analysed by means of Proposition 5.1

To put Theorem 7.1 into context, we note that the diagram in it participates in the following larger diagram:

$$\begin{array}{ccccc}
 \mathrm{KO}_*^G(\underline{EG}) & \xrightarrow{\mathrm{BC}} & & \mathrm{K}_*(C_r^*G) & \\
 \uparrow & & & \uparrow \simeq_{\geq 0} & \\
 \mathrm{ko}_*^G(\underline{EG}) & \xrightarrow{\quad\quad\quad} & & \mathrm{k}_*(C_r^*G) & \\
 \downarrow \tau & & & \downarrow & \\
 \mathrm{LR}_*^G(\underline{EG}) & \xrightarrow{\mathrm{FJ}_{\mathbb{R}}} & \mathrm{L}_*(\mathbb{R}G) & \longrightarrow & \mathrm{L}_*(C_r^*G) \\
 \uparrow & & \uparrow & & \\
 \mathrm{LZ}_*^G(\underline{EG}) & \xrightarrow{\mathrm{FJ}_{\mathbb{Z}}} & \mathrm{L}_*(\mathbb{Z}G) & &
 \end{array}$$

The map labelled τ in the diagram factors as

$$\mathrm{ko}_*^G(\underline{EG}) \longrightarrow \mathrm{LR}_*^G(\underline{EG}) \longrightarrow \mathrm{LR}_*^G(\underline{EG})$$

where the second map is split injective, see the argument below, and the first map is in principle understandable by means of Theorem A & B. We shall argue that the full Farrell–Jones conjecture for G implies that⁶

- (1) the map $\mathrm{FJ}_{\mathbb{R}}$ is an isomorphism, and
- (2) the map $\mathrm{FJ}_{\mathbb{Z}}$ is an isomorphism if G is torsion free.

To see statement (1) and the claim about the split injectivity above, we first note that there is a commutative diagram

$$\begin{array}{ccccc}
 \mathrm{LR}^G(\underline{EG}) & \longrightarrow & \mathrm{LR}^G(\underline{EG}) & \longrightarrow & \mathrm{L}(\mathbb{R}G) \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathrm{LR}^G(\underline{EG}) & \longrightarrow & \mathrm{LR}^G(\underline{EG}) & \longrightarrow & \mathrm{L}(\mathbb{R}G)
 \end{array}$$

and claim that the vertical maps are all equivalences: Indeed, the K-theoretic Farrell–Jones conjecture together with the fact that \mathbb{R} is a regular \mathbb{Q} -algebra and [LR05, Proposition 2.14] implies that $K(\mathbb{R}G)$ is connective, so that $\mathrm{L}(\mathbb{R}G) \rightarrow \mathrm{L}(\mathbb{R}G)$ is an equivalence, see [LR05, Conjecture 3.3]. To see that also middle and left vertical maps are equivalences, we use the same argument for G replaced by virtually cyclic subgroups and finite subgroups of G , respectively - note here that the class of group for which the (full) Farrell–Jones conjectures hold is closed under taking subgroups.

Now, the Farrell–Jones conjecture implies that the right lower horizontal map is an equivalence, and [LR05, Proposition 2.16] states that the left lower horizontal map is split injective.

Statement (2) requires different methods, since the Farrell–Jones conjecture is a conjecture about *quadratic* L-theory whereas we make a statement about *symmetric* L-theory. Note that this subtlety does not appear for group rings over \mathbb{R} since there, quadratic and symmetric L-theory agree. We give a proof of statement (2) in Theorem 7.6 below, relying on some recent developments in hermitian K-theory.

⁶That is, we assume that G is a Farrell–Jones group in the sense of [HLLRW21, §5.2].

The big diagram appearing above simplifies if the group G is torsion free: In this case one obtains the following diagram.

$$\begin{array}{ccccc}
BG \otimes KO & \xrightarrow{\text{BC}} & & \longrightarrow & K_*(C_r^*G) \\
\cong_{\geq \dim BG} \uparrow & & & & \uparrow \cong_{\geq 0} \\
BG \otimes ko & \longrightarrow & & \longrightarrow & k_*(C_r^*G) \\
\downarrow & & & & \downarrow \\
BG \otimes L(\mathbb{R}) & \xrightarrow{\text{FJ}_{\mathbb{R}}} & L(\mathbb{R}G) & \longrightarrow & L(C_r^*G) \\
\uparrow & & \uparrow & & \\
BG \otimes L(\mathbb{Z}) & \xrightarrow{\text{FJ}_{\mathbb{Z}}} & L(\mathbb{Z}G) & &
\end{array}$$

In addition, for torsion free groups, the Farrell–Jones conjecture in quadratic L-theory implies the one in symmetric L-theory, see the subsection below. However, the lower vertical comparison maps which change the base ring in the L-theoretic FJ conjecture from \mathbb{Z} to \mathbb{R} are quite subtle to analyse, in particular integrally, but even after inverting 2, see e.g. [LR05, Remark 3.20]. If furthermore BG has an n -dimensional classifying space, then the left top most vertical map is an equivalence in degrees $\geq n + 1$.

Farrell–Jones for symmetric L-theory. In this section we prove the following result we have indicated above and which might be of some independent interest.

Theorem 7.6. *Let G be a torsion free group and R a regular ring. Assume that the FJ conjecture holds for G . Then the assembly map*

$$BG \otimes L^s(R) \longrightarrow L^s(RG)$$

is an equivalence.

To connect it more explicitly to the statement (2) above, we also record here the following result.

Proposition 7.7. *Let R be a regular ring and G a torsion free group. Then the relative assembly map*

$$BG \otimes L^s(R) \longrightarrow L^s R^G(\underline{EG})$$

is an equivalence.

Proof. By the transitivity principle for assembly maps [LR05, Theorem 2.9], we need to show that for each virtually cyclic subgroup V of G , the assembly map $BV \otimes L^s(R) \rightarrow L^s(RV)$ is an equivalence. Now, since G is torsion free, so is V , and therefore V is either trivial or isomorphic to \mathbb{Z} [LR05, Lemma 2.15]. We therefore need to show that the *Shaneson–Ranicki splitting* holds for symmetric L-theory, which is for instance done in the generality of bordism invariant Verdier localising invariants of Poincaré categorie in [CDH⁺22], see [MR90] for an earlier proof of the splitting result for symmetric L-theory. Note that we also use that $K_0(RV) \cong K_0(R)$ in order to ensure that no decoration problems appear in the Shaneson–Ranicki splitting. \square

In what follows, we will freely make use of the language and notation developed in the sequence of papers [CDH⁺20a, CDH⁺20b, CDH⁺20c, CDH⁺22]. Suffice it to say here that for a space⁷ X and a Poincaré category $(\mathcal{C}, \mathcal{Q})$, we write

$$(\mathcal{C}, \mathcal{Q})_X = \text{colim}_X(\mathcal{C}, \mathcal{Q})$$

for the tensor of $(\mathcal{C}, \mathcal{Q})$ with X . We call $(\mathcal{C}, \mathcal{Q})_X$ the *visible* Poincaré category associated to X and $(\mathcal{C}, \mathcal{Q})$, see [CDH⁺20a] for some explanation of the terminology and how its L-theory connects to previously studied versions of visible L-theory. In the proof of the following result, which we initially learned from Yonatan Harpaz, we will describe the Poincaré category $(\mathcal{C}, \mathcal{Q})_X$ in more detail.

⁷here, best to be thought of as an ∞ -groupoid

Lemma 7.8. *Let X be a space, \mathcal{C} be a stable ∞ -category and $\mathcal{Q} \rightarrow \mathcal{Q}'$ a map of Poincaré structures on \mathcal{C} inducing an equivalence on the bilinear parts of \mathcal{Q} and \mathcal{Q}' . Then the diagram*

$$\begin{array}{ccc} X \otimes L(\mathcal{C}, \mathcal{Q}) & \longrightarrow & X \otimes L(\mathcal{C}, \mathcal{Q}') \\ \downarrow & & \downarrow \\ L((\mathcal{C}, \mathcal{Q})_X) & \longrightarrow & L((\mathcal{C}, \mathcal{Q}')_X) \end{array}$$

is a pullback, where the vertical maps are the assembly maps.

Proof. Let $T = \text{cofib}(\mathcal{Q} \rightarrow \mathcal{Q}')$, which is by assumption an exact functor $\mathcal{C}^{\text{op}} \rightarrow \text{Sp}$. It is therefore a filtered colimit of representables. All terms in the diagram appearing in the lemma preserve filtered colimits of Poincaré categories, so it suffices to prove the lemma in the case where T is represented by an object t of \mathcal{C} , i.e. where $T = \text{map}_{\mathcal{C}}(-, t)$. In this case, $T_X = \text{cofib}(\mathcal{Q}_X \rightarrow \mathcal{Q}'_X)$ is given as follows.

We recall that \mathcal{C}_X is the subcategory of $\text{Fun}(X^{\text{op}}, \text{Pro}(\mathcal{C}))$ ⁸ generated under finite limits from the right Kan extensions of functors $* \rightarrow \mathcal{C} \rightarrow \text{Pro}(\mathcal{C})$ along inclusions $* \rightarrow X^{\text{op}}$. We then have that for $\varphi \in \mathcal{C}_X \subseteq \text{Fun}(X^{\text{op}}, \text{Pro}(\mathcal{C}))$

$$T_X(\varphi) = \text{colim}_{x \in X} T(\varphi(x)) = \text{colim}_{x \in X} \text{map}(\varphi(x), t)$$

so that T_X is represented by the object $r^*(t)$ in $\text{Fun}(X^{\text{op}}, \mathcal{C}) \subseteq \text{Fun}(X^{\text{op}}, \text{Pro}(\mathcal{C}))$, where $r: X \rightarrow *$ is the unique map. Now in general, $r^*(t)$ is not contained in \mathcal{C}_X (though it is the case if X is compact to which the general case reduces again by using that all functors in sight preserve filtered colimits). Regardless, one can write it as a filtered colimit of objects in \mathcal{C}_X .

The formula for relative L-theory of [HNS22] says that there is an equivalence

$$L(\mathcal{C}; \mathcal{Q}, \mathcal{Q}') = \text{Eq}(\mathcal{Q}'(Dt) \rightrightarrows B_{\mathcal{Q}}(Dt, Dt)),$$

where D denotes the (common) duality of \mathcal{C} - here, one of the maps is the canonical forgetful map $\mathcal{Q}'(Dt) \rightarrow B_{\mathcal{Q}}(Dt, Dt)^{hC_2} \rightarrow B_{\mathcal{Q}}(Dt, Dt)$, and the other one is the canonical map $\mathcal{Q}'(Dt) \rightarrow B_{\mathcal{Q}}(Dt, Dt)^{hC_2} \rightarrow B_{\mathcal{Q}}(Dt, Dt)^{tC_2} \simeq \Lambda_{\mathcal{Q}^s}(Dt) \rightarrow \Lambda_T(Dt) = \text{map}_{\mathcal{C}}(Dt, t) \simeq B_{\mathcal{Q}}(Dt, Dt)$. Likewise, we obtain

$$L(\mathcal{C}_X; \mathcal{Q}_X, \mathcal{Q}'_X) = \text{Eq}(\mathcal{Q}'_X(D(r^*(t))) \rightrightarrows B_{\mathcal{Q}_X}(D(r^*(t)), D(r^*(t)))).$$

Now $D(r^*(t)) = r^*Dt$, and therefore, since $\mathcal{Q}_X(\varphi) = \text{colim}_X \mathcal{Q}(\varphi(x))$, and likewise for the bilinear functor [CDH⁺20a, Prop. 6.4.3], we find that

$$L(\mathcal{C}_X; \mathcal{Q}_X, \mathcal{Q}'_X) = \text{colim}_X \text{Eq}(\mathcal{Q}'(Dt) \rightrightarrows B_{\mathcal{Q}}(Dt, Dt)) = X \otimes L(\mathcal{C}; \mathcal{Q}, \mathcal{Q}')$$

and one checks that the maps are again the ones indicated above. The lemma then follows. \square

Recall that for a ring R with involution we have the stable ∞ -category $\mathcal{D}^p(R)$ of perfect complexes over R . The involution on R induces a canonical duality on $\mathcal{D}^p(R)$, giving rise to homotopy quadratic and homotopy symmetric Poincaré structures \mathcal{Q}^q and \mathcal{Q}^s which are related by the canonical symmetrisation map $\mathcal{Q}^q \rightarrow \mathcal{Q}^s$. This map is an equivalence on cross effects. Let us define, for ease of notation, for any space X , the visible symmetric and visible quadratic L-theory of X with coefficients in R as follows.

$$L^{\text{vs}}(X; R) = L((\mathcal{D}^p(R), \mathcal{Q}^s)_X) \quad \text{and} \quad L^{\text{vq}}(X; R) = L((\mathcal{D}^p(R), \mathcal{Q}^q)_X).$$

By analysing the linear part of the visible Poincaré structure, we find that there is a canonical map of Poincaré categories

$$(\mathcal{D}^p(R)_X, \mathcal{Q}^q) \rightarrow (\mathcal{D}^p(R), \mathcal{Q}^q)_X$$

is an equivalence, i.e. that visible quadratic L-theory is simply quadratic L-theory of the category $\mathcal{D}^p(R)_X$ with its induced duality; we will therefore also write $L^q(X; R)$ for $L^{\text{vq}}(X; R)$. We note that $\mathcal{D}^p(R)_X \subseteq \mathcal{D}^p(R[\Omega X])$ so that in total we obtain an equivalence $L^{\text{vq}}(X; R) \simeq L_c^q(R[\Omega X])$ and using the π - π -theorem, see e.g. [CDH⁺20c, Corollary 1.2.33 (i)], even a further equivalence⁹

⁸Of course, $X^{\text{op}} \simeq X$, but in order to get op's and colimits vs limits correct, it is best not to identify X with X^{op} just yet.

⁹under the assumption that X is connected and pointed.

$L_c^q(R[\Omega X]) \simeq L_c^q(R\pi)$ where $\pi = \pi_1(X)$ and the subscript c stands for an appropriate control, namely the one given by the image of the map $K_0(R) \rightarrow K_0(R\pi)$.

Corollary 7.9. *The diagram*

$$\begin{array}{ccc} X \otimes L^q(R) & \longrightarrow & X \otimes L^s(R) \\ \downarrow & & \downarrow \\ L^q(X; R) & \longrightarrow & L^{vs}(X; R) \end{array}$$

is a pullback.

Proof. This is a special case of Lemma 7.8. \square

The following is now the remaining piece in the proof of Theorem 7.6.

Lemma 7.10. *Let R be an involutive ring and G be a 2-torsion free group. Then there is a canonical equivalence*

$$L^{vs}(BG; R) \longrightarrow L_c^s(RG).$$

Here the subscript c stands for control in the subgroup $\text{Im}(K_0(R) \rightarrow K_0(RG)) \subseteq K_0(RG)$.

Proof. We first note that the visible symmetric Poincaré structure, for connected and pointed X , in one on $\mathcal{D}^p(R)_{BG} \subseteq \mathcal{D}^p(RG)$ where the subcategory is the one associated to the subgroup $\text{Im}(K_0(R) \rightarrow K_0(RG))$. Since Poincaré structures extend uniquely to idempotent completions, it suffices to argue that on the category $\mathcal{D}^p(RG)$, the visible Poincaré structure \mathcal{Q}^{vs} agrees with the \mathcal{Q}^s . Since the bilinear parts agree, it suffices to compare the linear terms. In this case, we have

$$\Lambda^{vs}(M) = \text{map}_{RG}(M, R^{tC_2}) \quad \text{whereas} \quad \Lambda^s(M) = \text{map}_{RG}(M, (RG)^{tC_2})$$

where RG has the C_2 -action given induced by the involution on R and the inversion action on G . The map from left to right is induced by the map $\{e\} \rightarrow G$. Now, as a module with C_2 -action, RG therefore decomposes according to the decomposition of G into transitive C_2 -sets as follows:

$$RG = \bigoplus_{g \in G[2]} R \oplus \bigoplus_{[g] \in G \setminus G[2]} \text{ind}_e^{C_2}(R).$$

In particular, if e is the only 2-torsion element in G , then the map $R \rightarrow RG$ induces an equivalence after applying $(-)^{tC_2}$. Therefore, in this case we find that the canonical map of Poincaré structures $\mathcal{Q}^{vs} \rightarrow \mathcal{Q}^s$ is an equivalence. \square

Proof of Theorem 7.6. We consider the following commutative diagram.

$$\begin{array}{ccc} BG \otimes L^q(R) & \longrightarrow & L^q(RG) \\ \downarrow & & \downarrow \\ BG \otimes L^q(R) & \longrightarrow & L^q(RG) \end{array}$$

The left vertical map is an equivalence since $K(R)$ is assumed to be connective. The lower horizontal map is the map which is predicted to be an equivalence by the FJ conjecture. The right vertical map is an equivalence if $K(RG)$ is connective (though this is *not* and if and only if). Now the K-theoretic FJ conjecture implies that $K(RG) \simeq BG \otimes K(R)$ which is again connective by assumption. We conclude that the top horizontal map is an equivalence. Now we use the pullback diagram

$$\begin{array}{ccc} BG \otimes L^q(R) & \longrightarrow & BG \otimes L^s(R) \\ \downarrow & & \downarrow \\ L^q(BG; R) & \longrightarrow & L^{vs}(BG; R) \end{array}$$

obtained in Corollary 7.9 together with the equivalences $L^q(BG; R) \simeq L^q(RG)$ (which holds for all groups G) and the equivalence $L^{vs}(BG; R) \simeq L^s(RG)$ of Lemma 7.10 (which holds for 2-torsion free groups G). We have argued above that the left vertical map is an equivalence, and therefore so is the right. \square

8. RELATIONS TO SIGNATURE GENERA

We now comment on a relation to previous approaches to comparing the Baum–Connes and Farrell–Jones assembly maps and thereby analytic and surgery theoretic approaches to the Novikov conjecture. We recall here that the Novikov conjecture is implied by either of the two assembly maps being rationally injective, and that [LN18, Theorem D] implies that the FJ assembly map is rationally injective if the BC assembly map is rationally injective. In several papers [HR05a, HR05b, HR05c, PS16, Wah13] maps from L-theory to $K(-)[\frac{1}{2}]$ -theory have been constructed in order to get such a comparison. The idea common to those approaches is to promote the signature operator of an oriented manifold to an appropriate K-theory class. We will review this operator below and connect it to our approach. Note, however, that by Theorem 9.3 no integral map from L-theory to K-theory exists and our maps $\tau_{\mathbb{R}}: ko \rightarrow L\mathbb{R}$ and $\tau_{\mathbb{C}}: ku \rightarrow L\mathbb{C}$ are indeed maps in the other direction.

The signature operator. Let us first review the signature operator, see e.g. [LM89, I.§6, Example 6.2]. To this end we let M be a closed, oriented, Riemannian manifold of dimension n . We consider the de Rham complex

$$\Omega^*(M; \mathbb{C}) = \bigoplus_{i=0}^n \Omega^i(M; \mathbb{C})$$

of complex valued differential forms on M with the operator $d: \Omega^*(M; \mathbb{C}) \rightarrow \Omega^*(M; \mathbb{C})$. The orientation and metric induce inner products on $\Omega^*(M; \mathbb{C})$ and we denote the formal adjoint of d by d^* as usual. We then consider the elliptic, first order differential operator $D_M := d + d^*$. With respect to the chiral grading defined below, D_M is called the *signature operator*. We have that

$$D_M^2 = D_M^* D_M = dd^* + d^*d =: \Delta_M$$

is the well known Laplace operator on M . Thus the solutions to $D_M = 0$ are given by the solutions to $\Delta_M = 0$ which are the harmonic forms. By Hodge theory, the harmonic forms are isomorphic to $\bigoplus H^*(M; \mathbb{C})$. Now we introduce the chiral $\mathbb{Z}/2$ -grading on $\Omega^*(M)$. This is not the grading by even and odd forms (with respect to which the operator $d + d^*$ is the Euler operator whose index is the Euler characteristic). Instead, the grading operator τ is defined on a p -forms ω by

$$\tau(\omega) = i^{p(p-1)+\lceil n/2 \rceil} * \omega$$

where $n = \dim M$ and $*$ is the Hodge- $*$ -operator. It is not hard to check that this is indeed a grading operator, i.e. that $\tau^2 = 1$.

Remark 8.1. One can also describe this picture using Clifford algebras as follows: one has a canonical isomorphism $\Omega^*(M; \mathbb{C}) \cong \Gamma(\text{Cliff}_{\mathbb{C}}(TM))$. Under this isomorphism the operator $d + d^*$ corresponds to the Dirac operator on the Clifford bundle $\text{Cliff}_{\mathbb{C}}(TM)$, see [LM89, I.§5, Example 5.9]. The chiral grading corresponds to the grading induced by multiplication with the complex volume element which is the section of $\text{Cliff}_{\mathbb{C}}(TM)$ given locally by $[n/2]e_1 \cdots e_n$ for an orthonormal frame e_1, \dots, e_n .

If n is even then τ anticommutes with D_M , so that in terms of the decomposition $\Omega^*(M; \mathbb{C}) = \Omega^*(M; \mathbb{C})^+ \oplus \Omega^*(M; \mathbb{C})^-$ the operator D_M restricts to $D_M^+ : \Omega^*(M; \mathbb{C})^+ \rightarrow \Omega^*(M; \mathbb{C})^-$. Then we find that the index

$$\begin{aligned} \text{ind}(D_M) &= \dim \ker (\Omega^*(M; \mathbb{C})^+ \xrightarrow{D_M^+} \Omega^*(M; \mathbb{C})^-) - \dim \text{coker} (\Omega^*(M; \mathbb{C})^- \xrightarrow{D_M^+} \Omega^*(M; \mathbb{C})^+) \\ &= \dim \ker (\Omega^*(M; \mathbb{C})^+ \xrightarrow{D_M^+} \Omega^*(M; \mathbb{C})^-) - \dim \ker (\Omega^*(M; \mathbb{C})^- \xrightarrow{D_M^-} \Omega^*(M; \mathbb{C})^+) \end{aligned}$$

is given by the signature of the manifold M , hence the name signature operator. By means of Kasparov's model of the K -homology of M given by $KU_0(M) \cong KK(C^0(M), \mathbb{C})$, we see that the operator D_M with respect to the chiral grading defines a class in $KU_0(M)$. In this picture taking the index corresponds to the pushforward $KU_0(M) \rightarrow KU_0(\text{pt}) = \mathbb{Z}$.

If n is odd, then τ commutes with D_M , and both τ and D_M anti-commute with the usual even/odd grading operator. We can then consider the operator D_M as graded via the even/odd

grading, and use τ to obtain in addition an action by $\mathrm{Cl}_{\mathbb{C}}(\mathbb{R})$ where the odd generator acts via τ . In this way, one obtains the signature operator of M as an element of

$$\mathrm{KU}_1(M) = \mathrm{KK}(C^0(M), \mathrm{Cl}_{\mathbb{C}}(\mathbb{R}))$$

see [RW06, pg. 49]. Finally we would like to bring the operators just constructed into the top degrees by multiplying with the bott class. To this end we note that we have that

$$\begin{aligned} \mathrm{KU}_0(M^{2k}) &\xrightarrow{\beta^n} \mathrm{KU}_{2k}(M^{2k}) = \mathrm{ku}_{2n}(M^{2k}) \\ \mathrm{KU}_1(M^{2k+1}) &\xrightarrow{\beta^n} \mathrm{KU}_{2k+1}(M^{2k+1}) = \mathrm{ku}_{2n+1}(M^{2k+1}) \end{aligned}$$

where the latter equalities holds since M is $2k$ and $(2k + 1)$ -dimensional, respectively.

Definition 8.2. For any n -dimensional closed oriented manifold M we define the class of the signature operator $[D_M]$ as the class in $\mathrm{ku}_n(M)$ just described.

One of the main goals of this section is to prove the following result. We denote by $\sigma_{\mathbb{C}}$ the composite

$$\mathrm{MSO} \xrightarrow{\sigma_{\mathbb{R}}} \mathrm{L}(\mathbb{R}) \longrightarrow \mathrm{L}(\mathbb{C})$$

of the Sullivan–Ranicki orientation with the canonical map induced by $\mathbb{R} \rightarrow \mathbb{C}$ and by $\tau_{\mathbb{C}}: \mathrm{ku} \rightarrow \mathrm{L}(\mathbb{C})$ the canonical map from Theorem A or [LN18]. Both maps induce maps on homology of M :

$$\mathrm{MSO}_n(M) \longrightarrow \ell(\mathbb{C})_n(M) \longleftarrow \mathrm{ku}_n(M)$$

again denoted by $\sigma_{\mathbb{C}}$ and $\tau_{\mathbb{C}}$, respectively. We denote by $[M] \in \mathrm{MSO}_n(M)$ the bordism class of the identity of M .

Proposition 8.3. *Let M be an n -dimensional closed oriented manifold. In the group $\ell(\mathbb{C})_n(M)$, we have the equality*

$$\tau_{\mathbb{C}}([D_M]) = 2^{\lfloor n/2 \rfloor} \cdot \sigma_{\mathbb{C}}([M])$$

up to 2-power torsion, that is, the difference between the two classes is a 2-power torsion element.

Before we explain how to prove this statement we would like to ask the following interesting and obvious question:

Problem 8.4. Does the equality of Proposition 8.3 hold integrally?

The proof of Proposition 8.3 will proceed in several steps. First note that it is enough to check that the elements agree in $\ell(\mathbb{C})[\frac{1}{2}]_n(M)$ which is what we will in fact do. We first translate the statement into homotopy theory. To this end we would like to understand the signature operator in terms of genera. First recall that for each map of graded rings $\Phi: \mathrm{MSO}_* \rightarrow R_*$ one can assign a Hirzebruch characteristic series

$$K_{\Phi}(t) = \frac{t}{\exp_{\Phi}(t)} \in (R_* \otimes \mathbb{Q})[[t]]$$

where $\exp_{\Phi}(t)$ is the inverse to the logarithm $\log_{\Phi}(t) = \sum_n \Phi(\mathbb{C}P^n) \frac{t^{n+1}}{(n+1)}$. If we give t degree -2 then the Hirzebruch series¹⁰ $K_{\Phi}(t)$ is of degree -2 . Note that since $\mathbb{C}P^n$ is nullbordant for odd n , the power series we consider here is really power series in t^2 . We will be interested in the cases $R_* = \mathrm{KO}_*$ and $R_* = \mathrm{KU}_*$. We can introduce the degree 0 element $z := \beta t \in \mathrm{KU}_*[[t]]$ and can rewrite the power series $K_{\Phi}(t)$ as a power series in z :

$$K_{\Phi}(z) \in \mathbb{Q}[[z]]$$

Even for the case KO this works since $z^2 = \beta^2 t^2$ exists in the rationalization (recall that $\beta^2 = x/2$, for $x \in \mathrm{KO}_4$ as considered earlier) and the power series is really a series in z^2 . We will thus also use this convention for KO and hope this does not lead to confusion. Proposition 8.3 is a consequence of the following more general result, as we will explain below.

¹⁰Topologically, the Hirzebruch series is the difference class in $H^0(\mathbb{C}P^{\infty}, R \otimes H\mathbb{Q})^{\times}$ between the orientation Φ and the standard rational orientation of $R \otimes H\mathbb{Q}$.

Theorem 8.5. (1) *There is a unique map of \mathbb{E}_∞ -rings $\mathcal{L}_{AS}: \text{MSO} \rightarrow \text{ko}[\frac{1}{2}]$ which on homotopy groups induces the map*

$$[M^{4n}] \mapsto 2^{-2n} \beta^{2n} \text{sign}(M^{4n}). \quad {}^{11}$$

(2) *For every space X , the induced map $\text{MSO}_*(X) \rightarrow \text{ko}[\frac{1}{2}]_*(X) \xrightarrow{c} \text{ku}[\frac{1}{2}]_*(X)$ takes a class*

$[M \xrightarrow{f} X]$ to $2^{-\lfloor n/2 \rfloor} f_([D_M])$ where $[D_M]$ is the signature class of Definition 8.2.*

(3) *The Hirzebruch characteristic series of \mathcal{L}_{AS} is given by*

$$K_{\mathcal{L}_{AS}}(z) = \frac{z/2}{\tanh(z/2)}.$$

(4) *We have a commutative diagram*

$$\begin{array}{ccc} \text{MSO} & \xrightarrow{\mathcal{L}_{AS}} & \text{ko}[\frac{1}{2}] \\ \sigma_{\mathbb{R}} \downarrow & & \downarrow \tau_{\mathbb{R}} \\ \ell(\mathbb{R}) & \xrightarrow{\text{can}} & \ell(\mathbb{R})[\frac{1}{2}] \end{array}$$

of \mathbb{E}_∞ -maps.

Remark 8.6. Before we prove this theorem, we note that the genus associated with \mathcal{L}_{AS} is not quite the ordinary signature genus since there are powers of 2 appearing. In fact, the characteristic series of the ordinary signature genus is $z/\tanh(z)$ by Hirzebruch's signature theorem. The genus we consider here first came up (to the best of our knowledge) in Atiyah-Singer's deduction of Hirzebruch's signature theorem using their index theorem, see [AS68], specifically in equation (6.5) on page 577 in loc. cit. and the discussion around it for the powers of 2 that appear. Therefore it is not surprising that this genus shows up here. Similar genera have also been considered by Sullivan to construct a version of the orientation $\sigma_{\mathbb{R}}$, see e.g. [MM79, §5.A]. Therefore, Theorem 8.5 might not come as a surprise to the experts. However, a highly structured statement as Theorem 8.5 (4) is only possible since we have also constructed the map $\text{ko}[\frac{1}{2}] \rightarrow \ell(\mathbb{R})[\frac{1}{2}]$ in a highly structured way.

Proof of Proposition 8.3. First, we note that the statement of Proposition 8.3 is equivalent to showing the claimed equality after inverting 2. By (4) of Theorem 8.5, we also have a commutative diagram of \mathbb{E}_∞ -ring spectra

$$\begin{array}{ccc} \text{MSO} & \xrightarrow{c\mathcal{L}_{AS}} & \text{ku}[\frac{1}{2}] \\ \downarrow \sigma_{\mathbb{C}} & & \downarrow \tau_{\mathbb{C}} \\ \ell(\mathbb{C}) & \xrightarrow{\text{can}} & \ell(\mathbb{C})[\frac{1}{2}] \end{array}$$

Thus, for an n -dimensional closed oriented manifold M , we have the commutative diagram

$$\begin{array}{ccc} \text{MSO}_n(M) & \xrightarrow{c\mathcal{L}_{AS}} & \text{ku}_n(M)[\frac{1}{2}] \\ \downarrow \sigma_{\mathbb{C}} & & \downarrow \tau_{\mathbb{C}} \\ \ell(\mathbb{C})_n(M) & \xrightarrow{\text{can}} & \ell(\mathbb{C})_n(M)[\frac{1}{2}] \end{array}$$

By (2) of Theorem 8.5, the top right composite sends $[M]$ to $2^{-\lfloor n/2 \rfloor} \tau_{\mathbb{C}}([D_M])$, so the commutativity of the above diagram indeed gives the equality

$$2^{\lfloor n/2 \rfloor} \cdot \sigma_{\mathbb{C}}([M]) = \tau_{\mathbb{C}}([D_M])$$

in $\ell(\mathbb{C})_n(M)[\frac{1}{2}]$ as claimed. \square

¹¹Informally, $\beta^{2n} \text{sign}(M^{4n})$ is the signature of M if n is even and is 2 times the signature if n is odd.

Proof of Theorem 8.5. We first prove the uniqueness statement involved in part (1). In fact we also prove the uniqueness result involving homotopy ring maps. To this end we consider the maps

$$\begin{aligned} \pi_0(\mathrm{Map}_{\mathbb{E}_\infty}(\mathrm{MSO}, \mathrm{ko}[\tfrac{1}{2}])) &\longrightarrow \pi_0(\mathrm{Map}_{\mathrm{Sp}}^{\mathrm{HoRing}}(\mathrm{MSO}, \mathrm{ko}[\tfrac{1}{2}])) \\ &\xrightarrow{\pi_*} \mathrm{Hom}_{\mathrm{Ring}}(\mathrm{MSO}_*, \mathrm{ko}[\tfrac{1}{2}]_*) \\ &\xrightarrow{K} \mathbb{Q}[[z]] \end{aligned}$$

where $\mathrm{Map}^{\mathrm{HoRing}}$ denotes the connected components of the space of maps of spectra that are homotopy ring maps and the first map simply forgets the \mathbb{E}_∞ -structure.

Our claim is that all these maps are injective. For the last map K the assertion is true since $\mathrm{ko}_*[\frac{1}{2}]$ injects into $\mathrm{KO}_* \otimes \mathbb{Q}$ and rationally, we can reconstruct a genus from its characteristic series. In order to show injectivity of the first two maps it suffices to show that the composites

$$\pi_0(\mathrm{Map}_{\mathbb{E}_\infty}(\mathrm{MSO}, \mathrm{ko}[\tfrac{1}{2}])) \rightarrow \mathbb{Q}[[z]] \quad \pi_0(\mathrm{Map}_{\mathrm{Sp}}^{\mathrm{HoRing}}(\mathrm{MSO}, \mathrm{ko}[\tfrac{1}{2}])) \rightarrow \mathbb{Q}[[z]]$$

are injective. This will follow from the theory of orientations developed in [May77] and further in [AHR10] as we explain now. First, we note that we can localize away from 2 and that the canonical map $\mathrm{MSpin}[\frac{1}{2}] \rightarrow \mathrm{MSO}[\frac{1}{2}]$ is an equivalence. We may therefore replace MSO above with MSpin . For any pair of \mathbb{E}_∞ -maps $f, g: \mathrm{MSpin} \rightarrow \mathrm{ko}[\frac{1}{2}]$ we have a difference map $f/g: \mathrm{bspin} \rightarrow \mathrm{gl}_1(\mathrm{ko}[\frac{1}{2}])$ which is a map of spectra. More precisely the space of \mathbb{E}_∞ -ring maps $\mathrm{MSpin} \rightarrow \mathrm{ko}[\frac{1}{2}]$ is a torsor over the space of spectrum maps $\mathrm{bspin} \rightarrow \mathrm{gl}_1(\mathrm{ko}[\frac{1}{2}])$. Similarly in the case of homotopy ring maps the difference map is an H -space map $\mathrm{bspin} \rightarrow \mathrm{gl}_1(\mathrm{ko}[\frac{1}{2}])$.

Moreover from the difference class f/g we can recover the quotient K_f/K_g since K_f was itself constructed rationally as a difference class of f with the standard rational orientation $\mathrm{MSO} \rightarrow H\mathbb{Q} \rightarrow \mathrm{ko}_\mathbb{Q}$. Thus the whole statement is implied by showing that the canonical maps

$$(8.1) \quad \pi_0 \mathrm{Map}_{\mathrm{Sp}}(\mathrm{bspin}, \mathrm{gl}_1(\mathrm{ko}[\tfrac{1}{2}])) \rightarrow \pi_0 \mathrm{Map}_{\mathcal{S}}^H(\mathrm{BSpin}, \mathrm{gl}_1(\mathrm{ko}[\tfrac{1}{2}])) \rightarrow \pi_0 \mathrm{Map}_{\mathcal{S}}^H(\mathrm{BSpin}, \mathrm{gl}_1(\mathrm{ko}_\mathbb{Q}))$$

are injective. Here, BSpin is the infinite loop space of the spectrum bspin . Since bspin is 3-connected, we note that all mapping spaces do not change when replacing $\mathrm{gl}_1(\mathrm{ko}[\frac{1}{2}])$ with $\tau_{\geq 1} \mathrm{gl}_1(\mathrm{ko}[\frac{1}{2}]) \simeq \tau_{\geq 1} \mathrm{gl}_1(\mathrm{ko})[\frac{1}{2}]$, and using connectedness of bspin again, we may replace $\mathrm{gl}_1(\mathrm{ko}[\frac{1}{2}])$ with $\mathrm{gl}_1(\mathrm{ko})[\frac{1}{2}]$ and similarly $\mathrm{gl}_1(\mathrm{ko}_\mathbb{Q})$ with $\mathrm{gl}_1(\mathrm{ko})_\mathbb{Q}$. Moreover, since H -space maps form a collection of connected components inside the space of all maps, we may also neglect the superscript H . The first of the two maps in (8.1) is then induced by the canonical map of spectra $\Sigma_+^\infty \mathrm{BSpin} \rightarrow \mathrm{bspin}$, the counit of the $(\Sigma_+^\infty, \Omega^\infty)$ -adjunction. We then consider the following fracture square pullback.

$$\begin{array}{ccc} \mathrm{gl}_1(\mathrm{ko})[\tfrac{1}{2}] & \longrightarrow & \prod_{p \neq 2} \mathrm{gl}_1(\mathrm{ko})_p^\wedge \\ \downarrow & & \downarrow \\ \mathrm{gl}_1(\mathrm{ko})_\mathbb{Q} & \longrightarrow & \left[\prod_{p \neq 2} \mathrm{gl}_1(\mathrm{ko})_p^\wedge \right]_\mathbb{Q} \end{array}$$

Mapping bspin and $\Sigma_+^\infty \mathrm{BSpin}$ into this pullback, we obtain pullback descriptions for both mapping spaces in question. We then observe that

$$\pi_1(\mathrm{Map}_{\mathrm{Sp}}(\mathrm{bspin}, \left[\prod_{p \neq 2} \mathrm{gl}_1(\mathrm{ko})_p^\wedge \right]_\mathbb{Q})) = 0 = \pi_1(\mathrm{Map}_{\mathrm{Sp}}(\Sigma_+^\infty \mathrm{BSpin}, \left[\prod_{p \neq 2} \mathrm{gl}_1(\mathrm{ko})_p^\wedge \right]_\mathbb{Q}))$$

This follows simply from the observation that the homotopy groups of the rational spectrum $\left[\prod_{p \neq 2} \mathrm{gl}_1(\mathrm{ko})_p^\wedge \right]_\mathbb{Q}$ are concentrated in degrees $4k$, the rational homotopy of bspin is in degrees $4k$ and BSpin has rational cohomology also concentrated in degrees $4k$. We deduce that for $X = \mathrm{bspin}$ and $X = \Sigma_+^\infty \mathrm{BSpin}$, the canonical map

$$\pi_0 \mathrm{Map}_{\mathrm{Sp}}(X, \mathrm{gl}_1(\mathrm{ko})[\tfrac{1}{2}]) \longrightarrow \pi_0 \mathrm{Map}_{\mathrm{Sp}}(X, \mathrm{gl}_1(\mathrm{ko})_\mathbb{Q}) \times \prod_{p \neq 2} \pi_0 \mathrm{Map}_{\mathrm{Sp}}(X, \mathrm{gl}_1(\mathrm{ko})_p^\wedge)$$

is injective. It therefore suffices to argue that the map $\Sigma_+^\infty \text{BSpin} \rightarrow \text{bspin}$ induces a π_0 injection upon mapping to $\text{gl}_1(\text{ko})_{\mathbb{Q}}$ and $\text{gl}_1(\text{ko})_p^\wedge$ for all $p \neq 2$ individually. Note that $\text{gl}_1(\text{ko})_{\mathbb{Q}} = \prod_{k \geq 1} H\mathbb{Q}[4k]$, so that it suffices to know that the map $\Sigma_+^\infty \Omega^\infty X \rightarrow X$ induces an injection on rational cohomology in all degrees, for all connective spectra X . To treat the p -adic case, we recall from [AHR10, Theorem 4.11], applied to KO_p^\wedge , that $\text{gl}_1(\text{KO})_p^\wedge \rightarrow L_{K(1)}\text{gl}_1(\text{KO}) \simeq \text{KO}_p^\wedge$ is 1-truncated. Using again that bspin is 3-connected, it suffices now to show that the map $\Sigma_+^\infty \text{BSpin} \rightarrow \text{bspin}$ induces a π_0 -injection on mapping spaces to KO_p^\wedge . Since this spectrum is $K(1)$ -local, it finally suffices to note that the map $\Sigma_+^\infty \text{BSpin} \rightarrow \text{bspin}$ has a $K(1)$ -local section. This is a consequence of the fact that $L_{K(1)} = \Phi\Omega^\infty$, where Φ is the Bousfield–Kuhn functor, see [LMMT20, proof of Prop. 2.9] for details. This shows that the first map of (8.1) is injective as claimed.

We turn to the second map of (8.1). Using again the fracture square for $\text{gl}_1(\text{ko})[\frac{1}{2}]$ as before, the statement follows if we can show that for F the fibre of the map

$$\prod_{p \neq 2} \text{gl}_1(\text{ko})_p^\wedge \longrightarrow \left[\prod_{p \neq 2} \text{gl}_1(\text{ko})_p^\wedge \right]_{\mathbb{Q}}$$

we have

$$\pi_0(\text{Map}_{\text{Sp}}(\Sigma_+^\infty \text{BSpin}, F)) = 0.$$

With a similar argument as before, using again that BSpin is 3-connected, we may equivalently replace F by the fibre of the map

$$\prod_{p \neq 2} \text{KO}^\wedge \longrightarrow \left[\prod_{p \neq 2} \text{KO}_p^\wedge \right]_{\mathbb{Q}}$$

which is, again by a fracture square argument the same fibre as that of the map $\text{KO}[\frac{1}{2}] \rightarrow \text{KO}_{\mathbb{Q}}$. Since this map is a retract of $\text{KU}[\frac{1}{2}] \rightarrow \text{KU}_{\mathbb{Q}}$, it finally suffices to show that

$$\pi_0(\text{Map}_{\text{Sp}}(\Sigma_+^\infty \text{BSpin}, \text{fib}(\text{KU}[\frac{1}{2}] \rightarrow \text{KU}_{\mathbb{Q}}))) = 0.$$

Since BSpin has even rational cohomology, this is equivalent to showing that the map

$$\pi_0(\text{Map}_{\text{Sp}}(\Sigma_+^\infty \text{BSpin}, \text{KU}[\frac{1}{2}])) \longrightarrow \pi_0(\text{Map}_{\text{Sp}}(\Sigma_+^\infty \text{BSpin}, \text{KU}_{\mathbb{Q}}))$$

is injective. For this, we recall that $\text{BSpin} \simeq \text{colim}_n \text{BSpin}(n)$ and that $\text{Spin}(n)$ is a compact connected Lie group. In [And69], Anderson shows that for any compact connected Lie group G , we have that $\text{KU} \otimes BG$ is a filtered colimit of direct sums of KU , with split injective transition maps. Consequently, we find that for any KU -module M , we have

$$\text{Map}_{\text{Sp}}(\Sigma_+^\infty \text{BSpin}(n), M) = \text{Map}_{\text{KU}}(\text{colim}_{i \in I} \bigoplus_{A_i} \text{KU}, M) = \lim_{i \in I} \prod_{A_i} M$$

naturally in M . In particular, the map

$$\text{Map}_{\text{Sp}}(\Sigma_+^\infty \text{BSpin}(n), \text{KU}[\frac{1}{2}]) \longrightarrow \text{Map}_{\text{Sp}}(\Sigma_+^\infty \text{BSpin}(n), \text{KU}_{\mathbb{Q}})$$

identifies with the map

$$\lim_{i \in I} \prod_{i \in A_i} \text{KU}[\frac{1}{2}] \longrightarrow \lim_{i \in I} \prod_{i \in A_i} \text{KU}_{\mathbb{Q}}$$

which is injective on π_0 . Finally, the map we wish to show is injective identifies with the map on π_0 induced by the map

$$\lim_n \text{Map}_{\text{Sp}}(\Sigma_+^\infty \text{BSpin}(n), \text{KU}[\frac{1}{2}]) \longrightarrow \lim_n \text{Map}_{\text{Sp}}(\Sigma_+^\infty \text{BSpin}(n), \text{KU}_{\mathbb{Q}}).$$

Now, since for each n both mapping spaces which appear have no odd homotopy groups, the $\lim\text{-}\lim^1$ -sequence shows that the map we wish to show is injective identifies with the map

$$\lim_n \pi_0 \text{Map}_{\text{Sp}}(\Sigma_+^\infty \text{BSpin}(n), \text{KU}[\frac{1}{2}]) \longrightarrow \lim_n \pi_0 \text{Map}_{\text{Sp}}(\Sigma_+^\infty \text{BSpin}(n), \text{KU}_{\mathbb{Q}}).$$

This is an inverse limit of injective maps, and hence itself injective as claimed. In particular, we have shown that there is at most one \mathbb{E}_∞ -map as in (1).

For the existence of this map, Ando–Hopkins–Rezk [AHR10, Theorem 6.1] give a concrete criterion in terms of certain p -adic congruences, see also [NS19, Theorem 3.1.1]. One could simply

verify these directly, which is for instance done by Wilson in [Wil15, Theorem 5.5]. Instead of using this calculation, we will proceed differently and simply use the square of part (4) as a proof of the existence of an \mathbb{E}_∞ -map with the correct effect on homotopy groups, since the right vertical map in it is an equivalence. This then also shows the commutativity of (4) immediately.

In order to prove statement (2), we use that the assignment

$$\mathrm{MSO}_n(X) \longrightarrow \mathrm{ku}[\frac{1}{2}]_n(X) \quad (M \xrightarrow{f} X) \mapsto 2^{-\lfloor n/2 \rfloor} f_*([D_M])$$

is a map of multiplicative cohomology theories as shown in [RW06], specifically see Remark 4 and Lemma 6 in loc. cit. Thus by the previous results it is enough to check that it agrees with the map of part (1) on coefficients, which is true by construction.

For (3) we want to compute the Hirzebruch series of the map \mathcal{L}_{AS} . We find that

$$\mathcal{L}_{AS}(\mathbb{C}P^n) = \begin{cases} 2^{-n} \beta^n & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

where $\beta = \beta_{\mathbb{C}}$ is the complex Bott element. Thus we get that

$$\begin{aligned} \log_{\mathcal{L}_{AS}}(t) &= t + \frac{\beta^2}{2^2} \cdot \frac{t^3}{3} + \frac{\beta^4}{2^4} \cdot \frac{t^5}{5} + \frac{\beta^6}{2^6} \cdot \frac{t^7}{7} + \dots \\ &= \frac{2}{\beta} \cdot ((\beta t/2) + \frac{(\beta t/2)^3}{3} + \frac{(\beta t/2)^5}{5} + \frac{(\beta t/2)^7}{7} + \dots) \\ &= \frac{2}{\beta} \tanh^{-1}(\beta t/2) \end{aligned}$$

The inverse of this power series (with respect to composition) is given by

$$\exp_{\mathcal{L}_{AS}}(t) = \frac{2}{\beta} \tanh(\beta t/2)$$

as one directly verifies. Therefore we get

$$K_{\mathcal{L}_{AS}}(t) = \frac{\beta t/2}{\tanh(\beta t/2)} = \frac{z/2}{\tanh(z/2)}$$

where we recall that $z = \beta t$. □

Remark 8.7. In [Wil15, Theorem 5.7], Wilson writes that in addition to the map $\mathcal{L}_{AS}: \mathrm{MSpin} \rightarrow \mathrm{ko}[\frac{1}{2}]$ described above, there also exists an integral map $\mathcal{L}_H: \mathrm{MSpin} \rightarrow \mathrm{ko}$ sending a spin manifold M^{4n} to $\beta^{2n} \mathrm{sign}(M)$. This, however, is a typo, and the map \mathcal{L}_H indeed only exists after inverting 2. The fact that it does exist after inverting 2 can be shown using the criterion of Ando–Hopkins–Rezk [AHR10, Theorem 6.1], or by postcomposing \mathcal{L}_{AS} with the Adams operation $\psi^2: \mathrm{ko}[\frac{1}{2}] \rightarrow \mathrm{ko}[\frac{1}{2}]$. We thank Johannes Sprang for explaining to us the following argument that the map does not exist at the prime 2. To explain this, we recall again the general result of Ando–Hopkins–Rezk: It says that the connected components of the space of \mathbb{E}_∞ -maps $\mathrm{MSpin} \rightarrow \mathrm{ko}$ are in bijection to the set of sequences¹² $(b_k)_{k \geq 2} \in \mathbb{Q}$ satisfying the following conditions:

- (1) $b_{2k+1} = 0$ for $k \geq 1$,
- (2) $b_{2k} \equiv -\frac{B_{2k}}{2k} \pmod{\mathbb{Z}}$, and
- (3) for every prime p and every element $c \in \mathbb{Z}_p^\times / \{\pm 1\}$, there exists a p -adic measure μ on $\mathbb{Z}_p^\times / \{\pm 1\}$ such that for all $k \geq 1$ one has¹³

$$(1 - p^{2k-1})(1 - c^{2k})b_{2k} = \int_{\mathbb{Z}_p^\times / \{\pm 1\}} x^{2k} d\mu(x)$$

The sequence relevant for realising the map sending M^{4n} to $\beta^{2n} \mathrm{sign}(M)$ as an \mathbb{E}_∞ -map $\mathrm{MSpin} \rightarrow \mathrm{ko}$ is the sequence

$$b_k := \frac{2^{k+1}(2^{k-1}-1)}{2k} B_k.$$

¹²For an \mathbb{E}_∞ -map $\mathrm{MSpin} \rightarrow \mathrm{ko}$, this sequence is given by the coefficients in the characteristic series as described above.

¹³We remark that $(1 - c^{2k})b_{2k}$ is a p -adic integer.

Conditions (1) and (2) above are indeed satisfied, see [Wil15, Proposition A.3] for (2) and (1) follows from the same property for the Bernoulli numbers B_k . Now, at prime 2, one observes that the sequence

$$(1 - 2^{2k-1})(1 - c^{2k})b_{2k}$$

converges to zero in \mathbb{Z}_2 . However, a sequence of moments, i.e. a sequence of the form

$$\int_{\mathbb{Z}_p^\times / \{\pm 1\}} x^{2k} d\mu(x)$$

converges to zero only if it is constantly zero. Indeed, $x \mapsto x^{2k+2^r} - x^{2k}$ for $x \in \mathbb{Z}_2^\times$ takes values in $2^r \mathbb{Z}_2$. Consequently,

$$\left| \int_{\mathbb{Z}_p^\times / \{\pm 1\}} x^{2k} d\mu(x) \right|_2 = \lim_{r \rightarrow \infty} \left| \int_{\mathbb{Z}_p^\times / \{\pm 1\}} x^{2k+\phi(2^r)} d\mu(x) \right|_2$$

where $|\cdot|_2$ denotes the 2-adic valuation, and the latter term is zero if we assume that the sequence of moments converges to zero. Now, the sequence we need to investigate converges to zero, but is not constantly zero, and is therefore not a sequence of moments.

We finish this section by noting that there is a commutative diagram

$$\begin{array}{ccccc} & & \mathcal{L}_{AS} & & \\ & \nearrow & & \searrow & \\ \text{MSO} & \xrightarrow{\mathcal{L}_H} & \text{ko}[\tfrac{1}{2}] & \xrightarrow{\psi^{-2}} & \text{ko}[\tfrac{1}{2}] \\ \downarrow \sigma_{\mathbb{R}} & & \downarrow \tau_{\mathbb{R}} & & \downarrow \tau_{\mathbb{R}} \\ \ell(\mathbb{R}) & \xrightarrow{\alpha} & \ell(\mathbb{R})[\tfrac{1}{2}] & \xrightarrow{\psi^{-2}} & \ell(\mathbb{R})[\tfrac{1}{2}] \\ & \searrow & \text{can} & \nearrow & \end{array}$$

where we denote by ψ^{-2} also the induced (inverse) Adams operation on $\ell(\mathbb{R})[\frac{1}{2}]$. The resulting map α is then given by the right-down composite in the diagram

$$\begin{array}{ccc} \ell(\mathbb{R}) & \xrightarrow{\text{can}} & \ell(\mathbb{R})[\tfrac{1}{2}] \\ \downarrow \psi^2 & & \downarrow \psi^2 \\ \ell(\mathbb{R}) & \xrightarrow{\text{can}} & \ell(\mathbb{R})[\tfrac{1}{2}] \end{array}$$

Question 8.8. Does there exist an \mathbb{E}_∞ -map $\psi^2: \ell(\mathbb{R}) \rightarrow \ell(\mathbb{R})$ rendering the above diagram commutative? Likewise, does there exist an \mathbb{E}_∞ -map $\psi^2: \ell(\mathbb{C}) \rightarrow \ell(\mathbb{C})$ rendering the analogous diagram

$$\begin{array}{ccc} \ell(\mathbb{C}) & \xrightarrow{\text{can}} & \ell(\mathbb{C})[\tfrac{1}{2}] \\ \downarrow \psi^2 & & \downarrow \psi^2 \\ \ell(\mathbb{C}) & \xrightarrow{\text{can}} & \ell(\mathbb{C})[\tfrac{1}{2}] \end{array}$$

commutative, where we use $\tau_{\mathbb{C}}: \text{ku}[\frac{1}{2}] \xrightarrow{\cong} \ell(\mathbb{C})[\frac{1}{2}]$ to define ψ^2 .

One can show that the map ψ^2 (in both the real and the complex case) exists as a map of \mathbb{E}_1 -algebras. In order to construct this, one can use that $\ell(\mathbb{R})$ and $\ell(\mathbb{C})$ are 2-locally the free \mathbb{E}_1 - $H\mathbb{Z}$ -algebra on a generator in degree 4 and 2, respectively, see also [HLN21, Corollary 4.2]. Then, the map ψ^2 is constructed as to be an $H\mathbb{Z}$ -algebra map at prime 2. At the time of writing, we do not know whether $\ell(\mathbb{R})$ or $\ell(\mathbb{C})$ are 2-locally \mathbb{E}_∞ - $H\mathbb{Z}$ -algebras, and in addition, should this be the case, we do not know whether to expect a possible \mathbb{E}_∞ -map $\psi^2: \ell(\mathbb{R}) \rightarrow \ell(\mathbb{R})$ to be 2-locally a map of \mathbb{E}_∞ - $H\mathbb{Z}$ -algebras.

Remark 8.9. A curious consequence of the existence of the \mathbb{E}_1 -map $\psi^2: \ell(\mathbb{C}) \rightarrow \ell(\mathbb{C})$ is the following observation about formal groups. We recall that the formal group of ku is the multiplicative one, in particular ku has a coordinate given by $x + y + \beta_{\mathbb{C}}xy$. The map $\tau_{\mathbb{C}}: ku \rightarrow \ell(\mathbb{C})$ provides a coordinate of the formal group of $\ell(\mathbb{C})$ which is then given by $x + y + 2b_{\mathbb{C}}xy$, where $b_{\mathbb{C}} \in L_2(\mathbb{C})$ is the periodicity generator, since $\tau_{\mathbb{C}}(\beta_{\mathbb{C}}) = 2b_{\mathbb{C}}$. Postcomposition with powers of ψ^2 on $\ell(\mathbb{C})$ gives another coordinate of the formal group of $\ell(\mathbb{C})$ given by $x + y + 2^k b_{\mathbb{C}}xy$, with $k \geq 1$. As any two coordinates of a formal group are connected by a (strict) isomorphism, we deduce that for $k \geq 1$, the formal group laws $x + y + 2xy$ and $x + y + 2^k xy$ are isomorphic over \mathbb{Z} .

9. FURTHER REMARKS

On maps between K-theory and L-theory. In this subsection, we aim to analyse, similarly to [LN18] the possible integral maps between K- and L-theory. Let us first consider the map $ko \rightarrow L(\mathbb{R})$ and describe its effect on homotopy groups. For this, and in general, it will be convenient to record the following result:

Lemma 9.1. *The transformation τ of Theorem A is compatible with the unique lax symmetric monoidal transformation τ of [LN18, Theorem A] in the sense that there is a commutative diagram of lax symmetric monoidal functors*

$$\begin{array}{ccc} k(-) & \xrightarrow{\tau} & L(-) \\ \downarrow & & \downarrow \\ k(- \otimes \mathbb{C}) & \xrightarrow{\tau} & L(- \otimes \mathbb{C}). \end{array}$$

Proof. One observes that the complexification functor sending A to $A_{\mathbb{C}} = A \otimes_{\mathbb{R}} \mathbb{C}$ from C^* -algebras to complex C^* -algebras descends to a symmetric monoidal functor

$$(-) \otimes \mathbb{C}: \mathrm{KK}_{\mathbb{R}} \longrightarrow \mathrm{KK}.$$

Then both composites of the diagram in question are lax symmetric monoidal transformations from $k(-) \rightarrow L(-)$. Using again that $k(-)$ is initial, there is (up to canonical equivalence) only one such transformation. Spelling this out explicitly, we obtain for each $A \in \mathbb{R}^* \mathrm{Alg}$ a commutative diagram

$$\begin{array}{ccc} k(A) & \longrightarrow & L(A) \\ \downarrow & & \downarrow \\ k(A_{\mathbb{C}}) & \longrightarrow & L(A_{\mathbb{C}}) \end{array}$$

which is natural in A . □

Example 9.2. Applying this in the case $A = \mathbb{R}$, we in particular obtain a commutative diagram of \mathbb{E}_{∞} -ring spectra given by

$$\begin{array}{ccc} ko & \xrightarrow{\tau} & L\mathbb{R} \\ \downarrow & & \downarrow \\ ku & \xrightarrow{\tau} & L\mathbb{C} \end{array}$$

and the map induced on homotopy rings of the lower horizontal map is given by

$$\mathbb{Z}[\beta] \longrightarrow \mathbb{Z}[b_{\mathbb{C}}]$$

sending β to $2b_{\mathbb{C}}$, see [LN18, Lemma 4.9]. Using this, we can again describe the map $ko \rightarrow L\mathbb{R}$ on homotopy rings as follows: Firstly, recall that

$$\pi_*(ko) = \mathbb{Z}[\eta, x, \beta_{\mathbb{R}}]/(\eta^3, 2\eta, \eta x, x^2 = 4\beta_{\mathbb{R}})$$

with $|\eta| = 1$, $|x| = 4$ and $|\beta_{\mathbb{R}}| = 8$. The map $ko \rightarrow ku$ vanishes on η , sends $\beta_{\mathbb{R}}$ to $\beta_{\mathbb{C}}^4$ and x to $2\beta_{\mathbb{C}}^2$. On homotopy, the map $L\mathbb{R} \rightarrow L\mathbb{C}$ identifies with the canonical inclusion

$$\mathbb{Z}[b_{\mathbb{C}}^2] \subseteq \mathbb{Z}[b_{\mathbb{C}}]$$

as the subring generated by $b_{\mathbb{C}}^2$. We denote the element in $L_4(\mathbb{R})$ corresponding to $b_{\mathbb{C}}^2$ by b . It then follows that the map $\text{ko} \rightarrow \text{L}\mathbb{R}$ sends x to $8b$ and $\beta_{\mathbb{R}}$ to $16b^2$. Notice that this is indeed compatible with the ring structure of $\pi_*(\text{ko})/\text{torsion}$ and our general analysis as in Proposition 5.1.

Just like in the complex case, the only possibility for an interesting integral map between K-theory and L-theory is the one just constructed. More precisely, the analog of [LN18, Theorem E] in the real case holds as well:

Theorem 9.3. *We have that*

$$[\text{KO}, \text{L}\mathbb{R}] = 0 = [\text{L}\mathbb{R}, \text{KO}] = [\ell\mathbb{R}, \text{KO}],$$

where the square brackets denote homotopy classes of maps.

Proof. The main ingredients in proving this result in the complex case are

- Both KU and LC are Anderson self-dual,
- the map $\text{KU} \otimes \text{LC} \rightarrow (\text{KU} \otimes \text{LC})[\frac{1}{2}]$ is an equivalence, and
- the spectrum $\text{KU} \otimes \text{LC}$ is even, i.e. has no odd homotopy groups.

The analog of these results hold true for KO in place of KU and $\text{L}\mathbb{R}$ in place of LC because

- $I_{\mathbb{Z}}(\text{KO}) \simeq \Sigma^4 \text{KO}$, see [HS14, Theorem 8.1] and $I_{\mathbb{Z}}(\text{L}\mathbb{R}) \simeq \text{L}\mathbb{R}$ simply because the homotopy groups of $I_{\mathbb{Z}}(\text{L}\mathbb{R})$ are again free of rank 1 over the homotopy groups of $\text{L}\mathbb{R}$, just like for LC .
- to show that 2 is invertible in $\text{KO} \otimes \text{L}\mathbb{R}$ it suffices to observe that $\text{KU} \simeq \text{cofib}(\eta: \Sigma \text{KO} \rightarrow \text{KO})$, and hence for any spectrum E in which η is trivial (such as $\text{L}\mathbb{R}$), we have

$$\text{KU} \otimes E \simeq \text{KO} \otimes E \oplus \Sigma^2 \text{KO} \otimes E.$$

It follows that $\text{KO} \otimes \text{L}\mathbb{R}$ is a direct summand in $\text{KU} \otimes \text{L}\mathbb{R}$ which is itself a direct summand of $\text{KU} \otimes \text{LC}$, as $\text{L}\mathbb{R} \oplus \Sigma^2 \text{L}\mathbb{R} \simeq \text{LC}$.

- We have just established that $\text{KO} \otimes \text{L}\mathbb{R}$ is a direct summand of $\text{KU} \otimes \text{LC}$, so is even as well.

□

Remark 9.4. We also remark that, as expected, $\ell(\mathbb{R})$ is not a perfect ko -module, and likewise that $\ell(\mathbb{C})$ is not a perfect ku -module. Indeed, it suffices to show the latter, as

$$\text{ku} \otimes_{\text{ko}} \ell(\mathbb{R}) \simeq \ell(\mathbb{C})$$

so if $\ell(\mathbb{C})$ is not perfect over ku , then $\ell(\mathbb{R})$ is also not perfect over ko . To show this, we observe that $\ell(\mathbb{C}) \otimes_{\text{ku}} \text{KU} = \text{KU}[\frac{1}{2}]$. It is obtained from $\ell(\mathbb{C})$ by inverting $2b$ and $L(\mathbb{C})[\frac{1}{2}] \simeq \text{KU}[\frac{1}{2}]$. Now, $\text{KU}[\frac{1}{2}]$ is not perfect over KU since it is not compact. Indeed, if it were compact we would have

$$\text{KU}[\frac{1}{2}] \simeq \text{map}_{\text{KU}}(\text{KU}[\frac{1}{2}], \text{KU}[\frac{1}{2}]) = \text{colim map}_{\text{KU}}(\text{KU}[\frac{1}{2}], \text{KU})$$

but this colimit is constant, as 2 is already invertible on the mapping space. The latter is then equivalent to $\lim \text{KU}$ with transition maps given by the multiplication by 2 map. But we have $\pi_0(\lim \text{KU}) = 0$ by the Milnor sequence.

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