Remarks on the norm of the U operator

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Let f be an overconvergent cusp form, and suppose f is r-overconvergent and has supremum norm 1 on the r-overconvergent locus. Then it is easily seen that $U(f)$ is pr-overconvergent; and it has supremum norm ≤ 1 on the r-overconvergent locus.

Question (from Kevin Buzzard, March 2006). *What is the maximum r' (depending on r) such that $U(f)$ must have norm ≤ 1 on the r'-overconvergent locus?*

In this short note, we answer this question completely in the case of weight 0, tame level 1 and $p \in \{2, 3, 5, 7, 13\}$. Here Kolberg's recurrences permit explicit calculations of the matrix of U:

Theorem 1. *There is a function f such that for any c with $|c| = p^{-12r/(p-1)}$, the functions $(cf, c^2 f^2, \dots)$ form a basis of $\mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z}), r)$, and the entries of the matrix of U in this basis are given by the formal power series expansion of $I_p(c^{-1}x, cy)$ where $I_p(x, y)$ is a rational function whose numerator and denominator have degree $p+1$, and whose denominator has constant term 1.*

It's clear that the matrix of U mapping from r_1-overconvergent to r_2-overconvergent forms is given by $I_p(c_2^{-1}x, c_1 y)$; U will have norm ≤ 1 if and only if this rational function has entries in $\mathbb{Z}_p$.

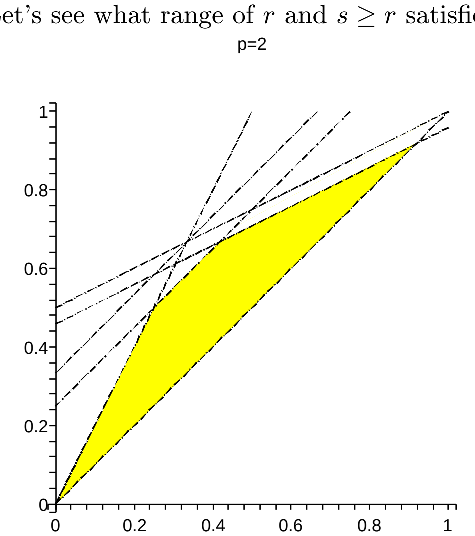
The program `umatrix_recurrence.gp` contains functions `mnorm(p, r1, r2)` and `snorm(p, r1, r2)` which calculate the matrices whose i, j entry is the valuation of the $x^i y^j$ term in the numerator and denominator respectively. Let us do the $p = 2$ case in detail. The recurrence matrix (corresponding to the denominator of I_p) is

$$\begin{pmatrix} 3 \times 2^{4+12r_1-12r_2} & 2^{24r_1-12r_2} \\ 2^{12+12r_1-24r_2} & 0 \end{pmatrix}$$

and the numerator matrix is

$$\begin{pmatrix} 3 \times 2^{3+12r_1-12r_2} & 2^{24r_1-12r_2} \\ 2^{11+12r_1-24r_2} & 0 \end{pmatrix}$$

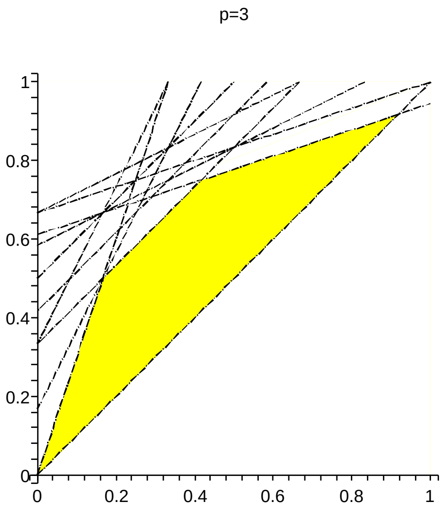
Let's see what range of r and $s \geq r$ satisfies the resulting six inequalities.



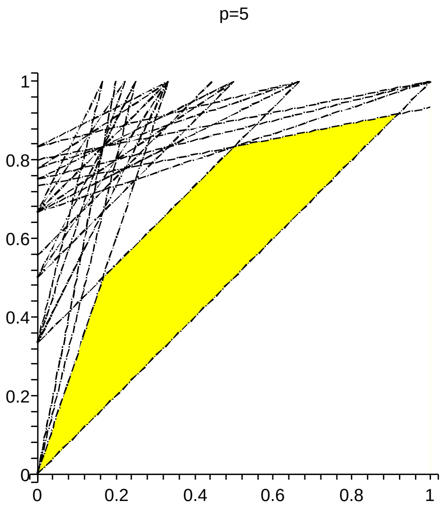
The feasible region is the closed quadrilateral with vertices at $(0,0)$, $(\frac{1}{4}, \frac{1}{2})$, $(\frac{1}{2}, \frac{5}{12})$, and $(\frac{11}{12}, \frac{11}{12})$.

So we can take $r' = 2r$ for any $r \leq \frac{1}{4}$. We can then stick with $r' = r + \frac{1}{4}$ for $\frac{1}{4} \leq r \leq \frac{5}{12}$. At this point we've got all the way up to $r' = \frac{p}{p+1}$-overconvergent; anything beyond this is a bit meaningless anyway, as it is no longer independent of our choice of lift of the Hasse invariant and so on, but we could go on up as far as $r = \frac{11}{12}$. One can visualise this in terms of lines bouncing alternately up and across inside the yellow region.

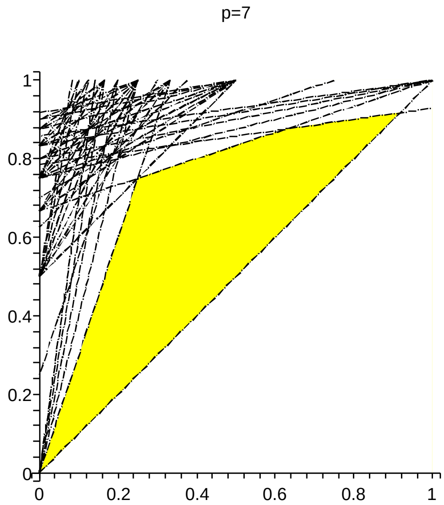
Now let's attack $p = 3$. In this case we get 12 inequalities, but most of them are not effective constraints, so we get this graph:



The allowed region has the same trapezoidal shape as before, so we may take $r' = 3r$ for $r \leq \frac{1}{6}$. Setting $r' = r + \frac{1}{3}$ then works up as far as $r = \frac{5}{12}$, and we've reached $\frac{3}{4}$-overconvergence.



For 5, we can't be quite as enthusiastic early on; $r' = 3r$ is the best we can do, as far as $r = \frac{1}{6}$. We now take $r' = r + \frac{1}{3}$ as far as $r = \frac{1}{2}$, by which point we've reached the target.



Here we start with $r' = 3r$ again, as far as $r = \frac{1}{4}$. But the next side has gradient $\frac{1}{3}$, and we must take $r' = \frac{1}{3}r + \frac{2}{3}$ as far as $r = \frac{5}{8}$, at which point we have arrived at $r = \frac{7}{8}$.

For $p = 13$, disaster! Nothing is integral for any $r' > r$ ever. This follows from the existence of a slope 0 eigenform.