Calculating p-adic modular forms

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Outline

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1. What is a p-adic modular form?
   - Motivation
   - Geometry of modular curves

The Eisenstein family

- Fix prime p
- Standard Eisenstein series

$$E_k = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n$$

- Easy to see that
  - $E_{p-1} = 1 \pmod{p}$
  - $E_{(p-1)p^r} = 1 \pmod{p^{r+1}} \quad \forall r \geq 0$
- Coefficients are p-adic analytic functions of $k \in (\mathbb{Z}/(p-1)\mathbb{Z}) \times \mathbb{Z}_p$
- Do cuspidal eigenforms also vary in p-adic families?

Serre's space

- Want this to be a *limiting process* (in some p-adic space)
- Serre: p-adic closure in $\mathbb{Z}_p[\![q]\!]$ of all $S_k(N)$, fixed N but arbitrary k
- Useful for studying p-adic L-functions (via constant terms of Eisenstein series)
- Space is *too big* – no good theory of eigenforms
- Katz: better construction, via line bundles on $X_0(N)$.

Outline

Modular forms and line bundles

- Recall $S_2(N)$ = differentials on $X_0(N)$ (holomorphic, or algebraic)
- More generally $S_k(N) =$ sections of ω^k, ω *Katz bundle*
- Can make $X_0(N)(\mathbb{Q}_p)$ into a "rigid space" (p-adic analytic geometry)
- Same answer in algebraic, holomorphic or rigid-analytic worlds ("GAGA")

Ordinary and supersingular loci

- Points of $X_0(N)(\mathbb{Q}_p)$ = elliptic curves over $\mathbb{Q}_p$ + level structure
- Bad behaviour at points where elliptic curve is *supersingular* mod p
- Serre's space = sections of ω^k over

$$X_0(N)_{ord} = X_0(N)(\mathbb{Q}_p) - \{\text{supersingular discs}\}$$

(a rigid space that is not algebraic).

Overconvergent modular forms

- Can "measure supersingularity" by valuation of E_{p-1}
- Rigid space

$$X_0(N)_{\geq r} = \{z \in X_0(N) : \mathrm{ord}_p(E_{p-1}(z)) \geq r\}$$

- Define $S_k^\dagger(N, r) =$ sections of ω^k over $X_0(N)_{\geq r}$.
- Independent of power of p dividing N

(More details: Emerton, *p-adic geometry of modular curves*)

These are not scary objects!

- Overconvergent modular forms:
  - have q-expansions (with p-adic coefficients)
  - action of Hecke operators (usual formulae)
  - Atkin-Lehner-Li theory (away from p)
  - include all modular forms of level Np^m (for small enough r)
- Interested in *Hecke eigenforms* in $S_k^\dagger(N, r)$
  - Good theory for those with $a_p \neq 0$ ("finite slope")
  - Always defined over finite extensions of $\mathbb{Q}_p$
  - p-adic L-functions, Galois representations, ...

Outline

1. What is a p-adic modular form?
2. How can we compute them?
3. Future directions

Outline

2 How can we compute them?

- Direct parametrisation of $X_0(N)$
- Koike's formula
- Overconvergent modular symbols

Direct parametrisation of $X_0(N)$

- Based on ideas of Emerton, Smithline, Buzzard-Calegari, Kilford, DL
- Proposition (Coleman): Any r-overconvergent form may be written as $E_{k-1} \times$ (p-adic analytic function on $X_0(N)_{\geq r}$)
- For small N and p, can just “write this down”:

$$\exists\, f : X_0(N)_{\leq r} \xrightarrow{\sim} \{x \in \mathbb{C}_p \ :\ |x| \leq p^{\varepsilon r}\}$$

- Hence

$$S_k^{\dagger}(N, r) = E_{k-1} \cdot \mathbb{Q}_p\langle p^{\varepsilon r} f\rangle$$

– just power series with a convergence bound

Hecke operators

- Choose f cuspidal $\Rightarrow$ basis has q-expansions in echelon form
- Use q-expansion formule to calculate Hecke matrices
- Can approximate small slope eigenforms based on truncations of matrix of T_p
- Implementation for $N = 1$, $p \in \{2, 3, 5, 7, 13\}$ in Sage since 3.4.1

Limitations

- At present need $k \in \mathbb{N}$ – easy to fix
- Generalisation to larger p, N difficult – need genus of $X_0(Np)$ small (preferably 0)
- Only 11 pairs (N, p) give genus 0
- Lloyd Kilford has something for genus 1 (in Magma)
- Clearly wrong approach for large levels

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2. How can we compute them?
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Koike's p-adic trace formula

Theorem (Koike)

For any fixed m, N, $p|m$, $(m, N) = 1$, have a formula

$$\mathrm{Tr}\left(T_m \mid S_k^\dagger(N, r)\right) = \sum_i \lambda_i \alpha_i^k$$

(for certain $\alpha_i \in \overline{\mathbb{Q}} \cap \mathbb{Z}_p^\times$).

- This is what you get if you take a "p-adic limit" in the Eichler-Selberg trace formula.

Classical and p-adic trace formulae

- Trace of T_5 on $S_k(\Gamma_0(3))$ (Eichler–Selberg–Hijikata):

$$\frac{2}{\sqrt{-11}}\left[\left(\frac{3+\sqrt{-11}}{2}\right)^{k-1}-\left(\frac{3-\sqrt{-11}}{2}\right)^{k-1}\right]-2(-5)^{\frac{k}{2}-1}-2.$$

- Take a careful 5-adic limit: $k_i \to \infty$, $k_i \to k$ 5-adically
- Trace of T_5 on 5-adic $S_k^\dagger(\Gamma_0(3), r)$ (for any r):

$$\frac{2}{\sqrt{-11}}\left(\frac{3+\sqrt{-11}}{2}\right)^{k-1}-2.$$

Traces and eigenvalues

- Formula gives trace of T_{p^t}, $\forall t > 0$
- Know $T_{p^t} = (T_p)^t$, so can calculate approximations to characteristic polynomial of T_p
- Similarly approximations to T_m, for $(m, N) = 1$, using $T_{p^t m}$
- Trace formula has many, many terms for large m

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Overconvergent modular symbols

- Due to Robert Pollack and Glenn Stevens
- Recall classical Manin symbols – basically pairs $[\gamma, f]$, $\gamma \in \Gamma\backslash \mathrm{SL}_2(\mathbb{Z})$, $f = f(X, Y)$ homogenous poly of degree $k - 2$, modulo certain relations
- Idea: replace f with a p-adic (locally) analytic function on $\mathbb{Z}_p$ (with an SL_2-action depending on k)
- Easier to work with dual (locally analytic distributions, isomorphic to a power series ring)

Pollack's implementation

- Robert Pollack has Magma code for this
- By-product of algorithm: can calculate p-adic L-functions rapidly
- Darmon + Pollack have used this to calculate Stark-Heegner points on elliptic curves

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3 Future directions
- Optimize linear algebra
- Other groups

Linear algebra

- Both direct parametrisation and modular symbols give *infinite* matrices $(m_{ij})_{i,j \in \mathbb{N}}$ (over $\mathbb{Q}_p$ or sometimes $\mathbb{Q}$)
- Convergence condition: $\mathrm{ord}_p(m_{ij}) \geq$ linear function of i, j
- Want to approximate small slope eigenvectors
- Need kernel algorithm that is “numerically stable”
- Pari’s algorithm seems to work well (better than native Sage or Magma)
- Interesting research area!

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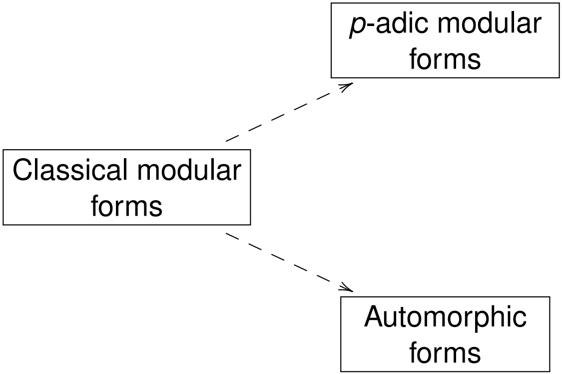
3 Future directions
- Optimize linear algebra
- Other groups

Automorphic forms on other groups

- Modular forms are *automorphic forms* for the *reductive group* $\mathrm{GL}_2(\mathbb{Q})$
- Can study automorphic forms for lots of other reductive groups, e.g.
  - $\mathrm{GL}_2(K)$, K totally real: Hilbert modular forms
  - $\mathrm{GSp}_{2n}(\mathbb{Q})$, $n \geq 1$: Siegel modular forms
  - $\mathrm{GL}_2(K)$, K imaginary quadratic: Bianchi modular forms
  - Unitary groups, orthogonal groups, E_8...
- For many of these groups, algorithms are known to compute automorphic forms

Two orthogonal generalisations



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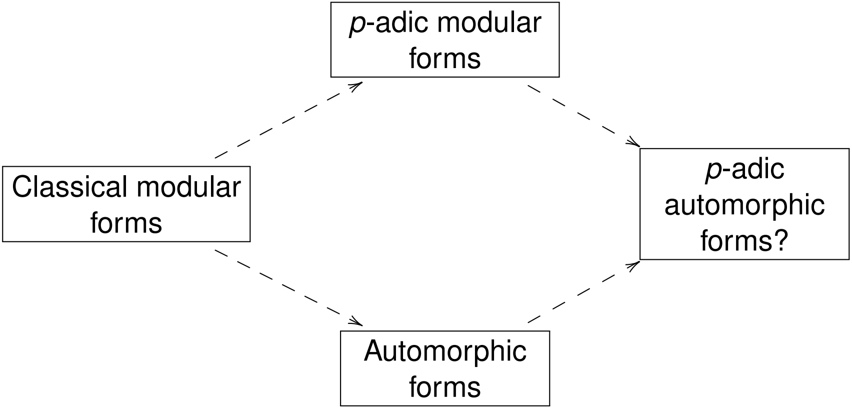

The future: p-adic automorphic forms?

- Should be a theory of p-adic automorphic forms for any reductive group
- General picture very unclear
- For some cases, have enough theory to compute something
- E.g. p-adic Hilbert modular forms: need Brandt modules over totally real fields
- Don't know yet what a "p-adic Bianchi modular form" should be