Some remarks on topological groups

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Problem. *Let G be a compact Hausdorff topological group. Show that the space $G/\sim$ of conjugacy classes is Hausdorff (in the quotient topology).*

The following definition will be useful to us:

Definition. *If a set $X \subset G$ is such that $g^{-1}Xg \subset X \ \forall g \in G$, we shall say that X is* ***normal****. The smallest normal set containing X shall be denoted $n(X)$, the* ***normal closure*** *of X.*

Lemma 1. *It is sufficient to prove that if $x, y \in G$ are not conjugate, there are open sets $U \ni x$, $V \ni y$ such that $U \cap V = \emptyset$ and U is normal.*

Proof. Let π be the projection map $G \to G/\sim$. Then the preimages under π of open sets in $G/\sim$ are clearly open and normal; conversely if a set $U \subset G$ is open and normal, $\pi^{-1}(\pi(U)) = U$, so $\pi(U)$ is open in $G/\sim$.

Hence the statement that $G/\sim$ is Hausdorff is equivalent to the statement that for any $x \not\sim y$, there are disjoint normal open sets $U \ni x$, $V \ni y$.

Now, if we assume that just one of the sets, U, is normal, suppose that $U \cap V = \emptyset$ but $U \cap n(V) \neq \emptyset$; let $z \in U \cap n(V)$. Then $z = g\zeta g^{-1}$ for some $g \in G$ and $\zeta \in V$; so $\zeta = g^{-1}zg$. Since U is normal it follows that $\zeta \in U \cap V$, contradiction. □

Lemma 2. *If X is any set and $\bar{X}$ is the topological closure of X, then $n(\bar{X}) = \overline{n(X)}$.*

Proof. On the one hand, the function $f(r, s) = rsr^{-1}$ is continuous $G \to G$, so if we restrict it to the compact set $G \times \bar{X}$ its range is compact. Hence $n(\bar{X})$ is closed, and since it obviously contains $n(X)$, we have $n(\bar{X}) \supset \overline{n(X)}$.

On the other hand, suppose $a \in n(\bar{X})$. Then $a = g^{-1}bg$ for some $b \in \bar{X}$. So $gag^{-1} \in C$ for every closed set $C \supset X$.

Hence in particular $gag^{-1} \in gCg^{-1}$ for every closed $C \supset n(X)$, since every such set is a closed set containing X.

So $a \in C$ for all closed $C \supset n(X)$; that is, $a \in \overline{n(X)}$. So $n(\bar{X}) \subset \overline{n(X)}$.

Therefore $n(\bar{X}) = \overline{n(X)}$. □

It's more or less downhill from here, if you know your separation axioms. Unfortunately, most of us don't, but I think this argument works.

Let U be any open set $\ni x$. If $y \notin \overline{n(U)}$, then $n(U)$ and $G - \overline{n(U)}$ separate x and y.

Suppose, therefore, that $y \in \overline{n(U)} \ \forall U \ni x$. We have

$$\begin{aligned} & y \in \overline{n(U)} \quad \forall U \ni x \\ \iff & y \in n(\bar{U}) \quad \forall U \ni x \\ \iff & [y] \cap \bar{U} \neq \emptyset \quad \forall U \ni x \end{aligned}$$

Now, it is sufficient to show that G is nice enough that given any closed set A and $x \neq A$, then there are open sets U, V with $x \in U$, $V \subset A$, and $\bar{U} \cap V = \emptyset$.

I claim this follows if G is a normal space:

Definition. *A topological space is* ***normal*** *if disjoint closed sets lie in disjoint open sets.*

For if G is normal, we may start with our closed sets being $\{x\}$ and A; we thus obtain U, V with $U \cap V = \emptyset$.

Now let our closed sets be $G - V$ and A; then we obtain sets U', V' such that $A \subset V$ and $\bar{U} \subset G - V \subset V'$; so U and V' have the required properties.

However, it is a known fact that any compact Hausdorff space is normal. So the result follows.