Norms of overconvergent weight 0 eigenforms

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The algorithm described in [Loe07] allows us to calculate approximations to the eigenfunctions of the U_p operator acting on overconvergent modular forms of tame level 1 for $p = 2, 3, 5, 7$ or 13. Here we shall use these to study how the sup norm $\|\phi_j\|_r$ of the jth eigenform ϕ_j on the overconvergent region $X_0(p)^{\leq r}$.

1 Variation with r

For each j, we may consider $\|\phi_j\|_r$ as a function of r, for $0 < r < \frac{p}{p+1}$. This will be an increasing, convex, piecewise-linear function of r.

We start with $p = 5$ and weight $k = 0$. The first cuspidal eigenfunction has slope 1, and its q-expansion is

$$\begin{aligned}\phi_1 = q + 8528631q^2 + 8596652q^3 + 2788848q^4 + 5 \times 610813q^5 + 6727787q^6 \\ + 2747331q^7 + 5 \times 3412617q^8 + 6989312q^9 + 5 \times 4155753q^{10} + O(q^{11}).\end{aligned}$$

For calculating norms, however, it is more convenient to express ϕ_1 as a power series in the usual uniformiser $f = (\eta(pz)/\eta(z))^3$: we have

$$\phi_1 = f + 5^3 \times 5849479f^2 + 5^5 \times 1026251f^3 + 5^8 \times 8671887f^4 + \dots$$

(where the coefficients are given in the form $5^a b$ where b is correct modulo 5^{10}). Suppose that $\phi_1 = \sum_{i\geq 0} b_i f^i$. Then, since $X_0(5)^{\leq r}$ is exactly the region where $|f| \leq p^{3r}$, we find that $\|\phi_1\|_r = p^{d(r)}$ where

$$d(r) = \sup_{i\geq 0} (3ri - \operatorname{ord}_p b_i).$$

Since we know that eigenforms are highly overconvergent, this supremum is finite for all $r < \frac{5}{6}$. We are not in a position to calculate it rigorously, but we can certainly calculate the supremum over the first few terms of the sequence and hope that this is the right answer!

The valuations of the first few b_i are apparently

$$0, 3, 5, 8, 10, 16, 19, 20, 23, 25 \dots,$$

so $d(r) = \sup\{3r, 6r - 3, 9r - 5, \dots\}$. In this case, the result is rather boring: the first term always dominates – for every $0 \leq r \leq \frac{5}{6}$, the norm is 5^{3r}. For the second eigenfunction ϕ_2, the b_i's have valuations

$$0, 0, 3, 5, 11, 10, 13, 14, 17, 19 \dots,$$

and it is the second term $6r$ that dominates, up until $r = \frac{7}{9}$ where the 8th term $(24r - 14)$ takes over.

For ϕ_3, the first term $3r$ has its brief moment in the sun, then the third term $9r - 1$ takes over. For $j = 10$, the tenth term eventually wins. I think that's the pattern right there. In fact f^j is the dominant term in f_j for these combinations of j's and r's:

$$\begin{aligned} j = 1 &: \quad \text{all r} \\ j = 2 &: \quad 0 \le r \le \frac{7}{9} \\ j = 3 &: \quad \frac{1}{6} \le r \le \frac{5}{6} \\ j = 4 &: \quad \frac{1}{6} \le r \le \frac{2}{3} \\ j = 5 &: \quad \frac{1}{12} \le r \le \frac{5}{6} \\ j = 6 &: \quad \frac{1}{3} \le r \le \frac{5}{6} \end{aligned}$$

Indeed a longer computation suggests that the jth term dominates the norm of f_j for a range of r which runs at least from $\frac{1}{3}$ to $\frac{2}{3}$. In the 3-adic setting, something similar happens, with the jth term dominating; interestingly, the range here is $\frac{1}{3}$ to $\frac{2}{3}$ again. The conjecture that the jth term dominates the norm of f_j in some open interval around $1/2$ is essentially Conjecture 3.1 in [Loe07].

I was convinced when I wrote the above that nothing so nice was going to happen for $p = 7$, but it seems we do still get the result that the governing term of the jth eigenfunction is f^j. So maybe the spectral expansion conjecture holds here, although there is less regularity in the sequence of slopes.

2 Variation with j

Now, let's fix r and consider $\|\phi_j\|_r$ as a function of j. The above suggests that $r = \frac{1}{2}$ is the most interesting value. Let's write $v_j = -\operatorname{ord}_p \|\phi_j\|_{1/2}$.

For $p = 5$, it seems that $v_j = \frac{1}{2}\sigma_j + \operatorname{ord}_5(j) + 1$:

j	v_j	σ_j
1	1.5	1
2	3	4
3	3.5	5
4	5	8
5	6.5	9
6	6	10
7	7.5	13
8	8	14
9	10.5	19
10	12	20
11	11.5	21
12	13	24
13	13.5	25

j	v_j	σ_j
14	15	28
15	16.5	29
16	16	30
17	18.5	35
18	19	36
19	20.5	39
20	22	40
21	21.5	41
22	23	44
23	23.5	45
24	25	48
25	27.5	49
⋮	⋮	⋮

For $p = 3$, this becomes $v_j = \frac{1}{2}\sigma_j + \operatorname{ord}_3(j) + 2$. For some reason my programs don't work so well for $p = 2$ at the moment, so I can't get anything useful in that case. For $p = 7$ the behaviour seems to be rather weird: from the first 20 values, it seems that $v_j = \frac{1}{2}\sigma_j + \epsilon_j$ where ϵ_j appears to be $\frac{1}{2}$ if n is odd and not divisible by 7, 1 if n is even and not divisible by 7, 1 if n is odd and divisible by 7, and 2 if n is divisible by 2 and 7. I can't work out the 49th value, but divisibility by higher powers of 2 seems to have no bearing.

Seemingly the only conclusion we can draw from this is that $v_j = \frac{1}{2}\sigma_j + \epsilon_j$ where ϵ_j is "some noise", but pinning down the noise might be more difficult.

References

[Loe07] David Loeffler, *Spectral expansions of overconvergent modular functions*, Int. Math. Res. Notices **2007** (2007), no. 050. MR 2353090