ADVENTURES WITH POLYNOMIALS: A CRITERION FOR WEIL NUMBERS

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Abstract. We consider the following problem: given a polynomial P over the complex numbers, when is it the case that all of the roots of P have the same absolute value?

1. Introduction

Recently, while doing some calculations related to automorphic forms on a definite unitary group, I encountered the polynomial

$$P(x) = x^3 - \frac{-7 + \sqrt{-259}}{8} x^2 + \frac{-7 - \sqrt{-259}}{4} x - 8.$$

A statement known as the generalised Ramanujan-Petersson conjecture implies that the roots of this polynomial should be *Weil numbers*: a Weil number is an algebraic integer all of whose Galois conjugates have the same absolute value. In this case, I expected that all of the roots of this polynomial should have absolute value 2. So I wondered: how would one actually check this? Of course, with today's computers it is a trivial matter to calculate the roots to any desired degree of accuracy; Maple opines that the roots are

$$\{1.606167789 + 1.191731945i, -0.7654506363 + 1.847724362i, \\ - 1.715717154 - 1.027771690i\}$$

and these have absolute values $1.999999999, 1.999999999, 2.000000001$. This looks pretty convincing; but how do we actually prove it?

More generally, I'll define a polynomial over $\mathbb{C}$ to be *good* if all of its roots have equal absolute value. So how do we tell if a polynomial is good?

2. A Trivial Case: Quadratics

Suppose we're given some quadratic polynomial, which we may as well suppose is monic, $x^2 + ax + b$. Then familiar A-level algebra tells us that the roots $\{\alpha, \beta\}$ satisfy $\alpha + \beta = -a$ and $\alpha\beta = b$.

We may as well re-scale everything by $\sqrt{b}$, and get a new polynomial $x^2 + a'x + 1$ where $a' = a/\sqrt{b}$; this will obviously be good if and only if the last one was. However, in this case it's clear: if the polynomial is good, then α and β must be complex conjugates of each other (since they are inverses of each other and have the same absolute value), so a' must be real and in the interval $[-2, 2]$, and the converse is clearly also true.

3. The Cubic Case

Let's try and repeat the same argument for cubics. First, we'll use scaling to reduce the problem to the special case where the constant term is 1, as before; so assume that we are given some general cubic polynomial $P(x) = x^3 + ax^2 + bx + 1$.

Lemma 1. *If P is good, then we must have $b = \overline{a}$.*

Proof. Let α, β, γ be the roots of P. Then the roots of $x^3+\overline{a}x^2+\overline{b}x+1$ are $\{\overline{\alpha}, \overline{\beta}, \overline{\gamma}\}$, and the roots of x^3+bx^2+ax+1 are $\{\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}\}$. If P is good, then these two sets are the same, so we must have $b = \overline{a}$. □

Now, we've reduced the entire problem to considering polynomials of the form $x^3 + ax^2 + \overline{a}x + 1$; and we know that for any polynomial of this type, the inverses of the roots are a permutation of their complex conjugates. Let S denote the subset of $\mathbb{C}$ for which this polynomial is good. How can we find a good description of S?

Let's take a little time out and think about the question intuitively. A monic polynomial over the complex numbers is uniquely specified by its roots – hence, 3 complex parameters, or 6 real ones. The subspace corresponding to roots of polynomials of the form $x^3 + ax^2 + \overline{a}x + 1$ only has real dimension 2, though, since we're only allowed to choose the real and imaginary parts of a. On the other hand, how many ways can I choose three points α, β, γ on the unit circle, subject to the requirement that $\alpha\beta\gamma = -1$? I can choose the arguments of α and β arbitrarily, and for any choice of these two numbers there is a unique γ that works. So the subset S of the a-plane which works is also 2-dimensional. Also, it is connected, and compact – that is, closed and bounded – because I could only choose my α and β from a connected compact set. So we've now changed the question into a geometric one, about describing a certain locus in the plane.

Indeed, if we let $\alpha = e^{i\theta}$ and $\beta = e^{i\phi}$, we are forced to take $\gamma = e^{-i(\theta+\phi)}$, and since $a = -(\alpha + \beta + \gamma)$ it follows that:

Lemma 2. *The set S is exactly the set of points which may be written in the form* $e^{-i(\theta+\phi)} - e^{i\theta} - e^{i\phi}$.

Note that this equals $-e^{i\theta} - 2ie^{-i\theta/2}\sin(\phi + \theta/2)$, so the set we're looking for is a union of straight line segments, each of length 4, corresponding to fixing the value of α. We can equally well choose a value of β or of γ, so each point on the set naturally lies on 3 lines.

Clearly the ends of these lines are boundary points; so the boundary of our set at least contains the curve in the plane described by $-e^{i\theta} \pm 2e^{i\theta/2}$. With a bit of imagination, one can visualise this as the image of a marked point on a circle of radius 1 rolling around the inside of a circle of radius 3 centred at the origin in the Argand plane, with the point starting at -3. This locus is a classical curve called a *deltoid*. Conversely, it's clear that the lines fill in the whole interior of the deltoid; see Figure 1.

Note that every point inside the set is on three lines, and each point on a line corresponds to two values of θ; this gives the six-fold symmetry, corresponding to the 6 ways of ordering the roots of the cubic. The diagram has another sort of six-fold symmetry in that if a is possible, then $\overline{a}$ is, and so is λa for λ each of the three cube roots of 1; so reflection in the x-axis and rotations by $\frac{2\pi}{3}$ preserve it. Points on the edges correspond to cubics with double roots, and the cusps correspond to the three values of a for which $x^3 + ax^2 + \overline{a}x + 1$ has a triple root (the three cube roots of -27).

4. An Algebraic Formula

This picture is very pretty, but it does not completely solve the problem: how do we verify whether or not a given value of a lies inside the deltoid? Here we use the fact that the boundary of S corresponds to good polynomials with a double root, and hence discriminant zero. The discriminant of $T^3 + (x+iy)T^2 + (x-iy)T + 1$ is easily checked to be

$$C(x, y) = (x^2 + y^2)^2 - 8x(x^2 - 3y^2) + 18(x^2 + y^2) - 27;$$

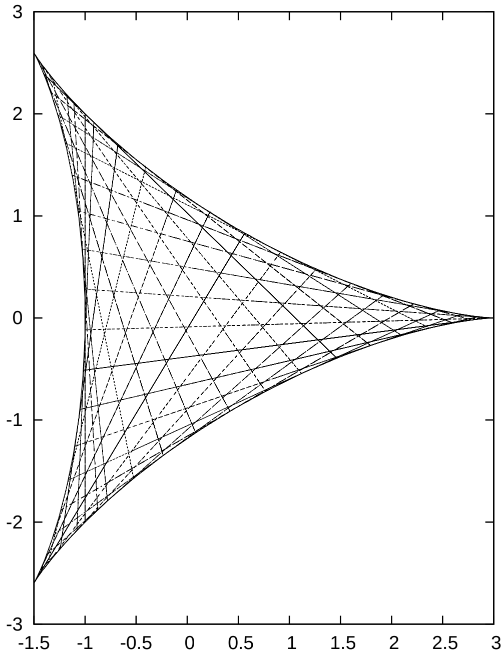

FIGURE 1. The solution set S for cubic polynomials

and it's easily seen that the set S corresponds to $C(x, y) \leq 0$, and the rest of the Argand plane to $C(x, y) \geq 0$.

Let's verify this for my pet example above. We started with the cubic

$$x^3 - \frac{-7 + \sqrt{-259}}{8}x^2 + \frac{-7 - \sqrt{-259}}{4}x - 8.$$

Scaling x by a factor of -2, we get a polynomial of the form above, $x^3 + ax^2 + \overline{a}x + 1$, for

$$a = \frac{-7 + \sqrt{-259}}{16}.$$

This corresponds to the point $(x, y) = \left(-\frac{7}{16}, \frac{\sqrt{259}}{16}\right)$ in the (x, y) plane. We find that

$$C\left(-\frac{7}{16}, \frac{\sqrt{259}}{16}\right) = -\frac{56727}{4096} \leq 0$$

so, as required, the original polynomial was good, with three roots of equal absolute value 2.

5. Quartics and beyond

For a general monic polynomial $P(x) = x^d + \sum_{i=1}^{d} a_i x^{d-i}$ of degree d, we can certainly still scale so that $a_d = 1$, and the argument of Lemma 1 implies that if P is good, the coefficients must satisfy $a_{d-i} = \overline{a_i}$. In particular, if d is even then $a_{d/2}$ is real. So the set S naturally lives in $\mathbb{R}^{d-1}$, and as before it is connected, compact, and has dimension equal to $d - 1$.

The projection of S onto $\mathbb{R}^2$ that corresponds to the real and imaginary parts of a_1 will be the obvious generalisation of the deltoid, a hypocycloid of d cusps. Seeing the effect of the further coordinates is more difficult; a 2-D projection of the 3-dimensional solution set for quartic polynomials is shown in Figure 2 below.

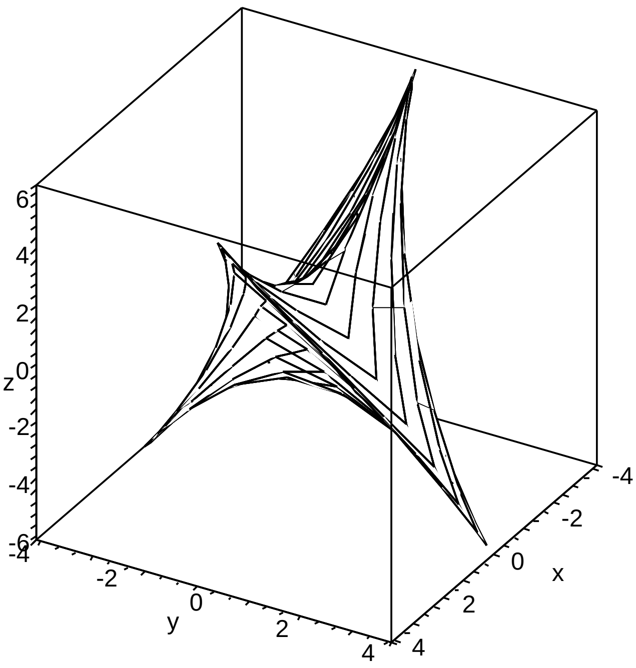

FIGURE 2. The solution set for quartics: x and y axes are the real and imaginary parts of the complex number a_1, and z axis is the real number a_2.