Slopes of 3-adic overconvergent modular forms

David Loeffler

May 2008

1 Introduction

This note concerns the properties of the Atkin-Lehner U operator acting on 3-adic overconvergent cusp forms. We begin by recalling the standard choice of basis, as described in [Loe07] or [Smi00].

Let $p \in \{2, 3, 5, 7, 13\}$. Then the space of r-overconvergent p-adic cuspidal modular forms has a basis $\{g, g^2, g^3, \dots\}$, where

$$g = p^{\frac{12}{p-1} r} \left(\frac{\Delta(pz)}{\Delta(z)} \right)^{\frac{1}{p-1}}.$$

If $r = \frac{1}{2}$ and (u_{ij}) is the matrix of the U operator in the above basis, then the numbers u_{ij} satisfy a recurrence formula: there is a $p \times p$ matrix M such that $u_{ij} = \sum_{r,s=1}^{p} M_{rs} u_{i-r,j-s}$. Furthermore, M is skew-upper-triangular and constant on off diagonals; and the coefficients u_{ij} satisfy $u_{ij} = \frac{j}{i} u_{ji}$.

The case $p = 2$ is extensively studied in [BC05]. Here the recurrence relation is simple enough that one can explicitly write down u_{ij} as a hypergeometric term in (i, j) and verify that it satisfies the correct initial conditions and recurrence. One can similarly spot a hypergeometric formula for the terms of the unique matrices A, D, B such that $ADB = U$ and A is lower triangular, B is upper triangular, A and B have diagonal entries 1 and D is diagonal; that these formulae give the correct entries for U can be checked by evaluating the corresponding sum using Dougall's ${}_7F_6$ identity.

From the formulae for A and B when $p = 2$, one can easily check that A and B are congruent to the identity matrix mod p; it follows that the slopes of U are equal to the valuations of the terms of D. One can also use this to prove the Gouvea-Mazur spectral expansion conjecture (that the finite slope eigenfunctions span the space), as shown in [Loe07].

For $p = 3$ things are more complicated. In [Loe07, Conjecture 3.1], it was noted that the entries of the matrix D appear to satisfy

$$D_{ii} = \frac{3^{3i}(6i)!(2i)!i!}{2 \cdot (3i)!^3},$$

and that the entries of A and B appear to be congruent to the identity for all $r \in \left(\frac{1}{3}, \frac{2}{3}\right)$. The results of [Loe07] then imply that the ith slope is equal to $\nu_3 D_{ii} = 2i + 2\,\nu_3 \left(\frac{(2i)!}{i!}\right)$ (for any r), and that the Gouvea-Mazur spectral expansion conjecture holds for $r \in \left(\frac{1}{3}, \frac{2}{3}\right)$.

In this note, we present some partial results towards proving this conjecture using Doron Zeilberger's theory of *holonomic* recurrence relations, as in [Zei90].

2 Holonomicity for U

We shall recall from [Zei90] some basic definitions and properties of holonomic functions.

A function a on $\mathbb{N}^n$ is said to be *holonomic* if for each variable i_r, $r = 1 \dots n$, it satisfies a recurrence relation in i_r with polynomial coefficients,

$$\sum_{t=1}^{m} P_t(i_1, \dots, i_n) a(i_1, \dots, i_r + t, \dots, i_n) = 0,$$

and the polynomials P_t are such that the space of functions satisfying these relations is *finite-dimensional*. Note that a holonomic function is specified by a finite amount of data – the coefficients of the polynomials in the recurrence, and a finite number of initial conditions.

Holonomic functions are closely related to the Weyl algebra $\mathcal{A}_n(k)$; there is a natural action of this algebra on functions on $\mathbb{N}^n$, and holonomic functions are those that generate a $\mathcal{A}_n(k)$-module of minimal Hilbert dimension, namely n (a holonomic $\mathcal{A}_n(k)$-module). Holonomic functions have many good properties: for example, sums and products of holonomic functions are holonomic, and evaluating at a fixed value of one variable or summing over all values of that variable send holonomic functions on $\mathbb{N}^n$ to holonomic functions on $\mathbb{N}^{n-1}$.

Furthermore, if P/Q is a rational function of variables $x_1, \dots, x_n$ such that Q has nonzero constant term (so the formal power series expansion of P/Q makes sense), then the coefficient of $x_1^{i_1} \dots x_n^{i_n}$ of the formal power series of P/Q is a holonomic function of $i_1, \dots, i_n$.

In [Loe07], following [Smi00], it was shown that the entries u_{ij} of the matrix of the U operator acting on the space of 3-adic $1/2$-overconvergent cusp forms of tame level 1 and weight 0, with respect to a certain standard basis, are equal to the coefficients of the formal power series expansion of a certain rational function $F(x, y)$; that is, we have

$$F(x,y) = \frac{90xy + 324x^2y + 648xy^2 + 243x^3y + 486x^2y^2 + 729xy^3}{1 - 270xy - 972xy(x+y) - 729xy(x^2+xy+y^2)} = \sum_{i,j=1}^{\infty} u_{ij}x^iy^j.$$

Thus u_{ij} is a holonomic function of two variables, by the above lemma. One can explicitly write down the recurrences obeyed by u_{ij}. The recurrence in i turns out to be

$$\begin{aligned}27(3j-i)(i+j)(i+j+1)u_{ij} + 18(j^2+6ij+8j-3i^2-6i-3)(i+j+1)u_{i+1,j}\\ +(j-4-3i)(j-5-3i)(j-6-3i)u_{i+2,j} = 0.\end{aligned} \qquad (\dagger)$$

Verifying that u_{ij} does indeed satisfy this relation, and the corresponding recurrence in j, is routine, as the recurrence relation for the coefficients translates into a differential equation for the generating function; this could in principle be checked by hand, but it is vastly easier to use a computer algebra package.

3 Factorisations for U

Let us define the following quantities:

$$\begin{aligned}
r_m(i) &= \prod_{\substack{t=1\dots 3m\\ 3\nmid t}} (3i+t) = \frac{(3i+3m)!i!}{3^m(3i)!(i+m)!}\\
\mu(i,j,t) &= \frac{2ji!j!(2i+2t)!(2i+t-1)!}{(i+j)!(i-j+2t)!(2i)!(j-i-t)!(i-t)!(i+t)!}\\
B_{ij} &= \sum_t \frac{3^{2t}}{r_t(i)}\mu(i,j,t)\\
A_{ij} &= \frac{j}{i}B_{ji}\\
D_{ij} &= \frac{3^{3i}(6i)!(2i)!i!}{2\cdot(3i)!^3} \quad \text{if } i=j, 0 \text{ otherwise}
\end{aligned}$$

When any of the factorials in the denominator are negative, $\mu_{i,j,t}$ is to be interpreted as zero, so the summation for B is in fact over t between $\frac{j-i}{2}$ and $\min(j-i, i)$; in particular, the summation is empty unless $i \le j$, so the matrix B is upper-triangular, and A is thus lower-triangular. It is easily seen that the diagonal entries in both cases are 1.

3.1 Lemma. *We have $B_{ij} \in 3^{j-i}\mathbb{Z}_3$, and $A_{ij} \in 3^{i-j}\mathbb{Z}_3$.*

Proof. We shall prove the stronger result that $B_{ij}/j \in 3^{j-i}\mathbb{Z}_3$ as long as $j > i$, from which both statements evidently follow. Since terms in the sum for which $t \le \frac{j-i}{2}$ are zero, it is enough to show that $\mu(i,j,t)/j \in \mathbb{Z}$.

Note that we must have $t \ge 1$ if $i < j$, so we may write

$$\frac{\mu(i,j,t)}{j} = 2\left(\frac{(2i+2t)!}{(i-j+2t)!(i+j)!}\right)\left(\frac{(2i+t-1)!}{(2i)!}\right)\left(\frac{j!}{(j-i-t)!(i+t)!}\right)\left(\frac{i!}{(i-t)!}\right)$$

and every bracket is clearly an integer. □

Conjecture. *Regarding A, D and B as matrices in the obvious way, we have $ADB = U$.*

Let us write $b_{ij}^{(t)} = \frac{3^{2t}}{r_t(i)}\mu(i,j,t)$, so that $b_{ij} = \sum_t b_{ij}^{(t)}$, and correspondingly $a_{ij}^{(t)} = \frac{j}{i}b_{ij}^{(t)}$. So the conjecture is that

$$\sum_{\alpha,\beta,\gamma} a_{i\gamma}^{(\alpha)} d_{\gamma\gamma} b_{\gamma j}^{(\beta)} = u_{ij}$$

for each i, j.

If we write $\Lambda(i,j,\alpha,\beta,\gamma)$ for the summand, then Λ is a holonomic function of $i, j, \alpha, \beta, \gamma$; and consequently $\sum_{\alpha,\beta,\gamma} \Lambda_{i,j,\alpha,\beta,\gamma}$ is a holonomic function of i, j. We must verify that the recurrences it satisfies are the same as those that are satisfied by u_{ij}, and that it satisfies the same initial conditions.

4 An application of Zeilberger's method

Let's write $b_{ij}^{(t)} = \frac{3^{2t}}{r_t(i)}\mu_{ijt}$, so that $b_{ij} = \sum_t b_{ij}^{(t)}$, and similarly $a_{ij}^{(t)}$. If we define

$$\gamma_{i,j,s,t,p} = a_{ip}^{(s)} d_p b_{pj}^{(t)},$$

then the conjecture will follow if we can show that

$$\sum_{s,t,p} \gamma_{i,j,s,t,p} = u_{ij}$$

for all i, j. Let us write v_{ij} for the left-hand side, so we conjecture that $v_{ij} = u_{ij}$.

Zeilberger's method of creative telescoping [PWZ96] shows that $\gamma_{i,j,s,t,p}$ satisfies a relation of the form

$$\mathcal{O}_i \gamma_{i,j,s,t,p} = \phi_{i,j,s+1,t,p} - \phi_{i,j,s,t,p}$$

where ϕ is an auxilliary function and $\mathcal{O}_i$ is a degree 2 shift operator in i, with coefficients that are polynomials independent of s. Fortuitously, they are also independent of t, so we can sum over all s and all t to obtain

$$\mathcal{O}_i v_{ij}^{(p)} = 0.$$

Here

$$v_{ij}^{(p)} = \sum_{s,t} \gamma_{i,j,s,t,p} = a_{ip} d_p b_{pj}.$$

Similarly there exists a shift operator $\mathcal{O}_j$ in j annihilating $v_{ij}^{(p)}$. These operators are

$$\begin{aligned}\mathcal{O}_i v_{ij}^{(p)} = 9(i+1)(i+2)(i+3p)(i-3p)v_{ij}^{(p)} \\ + 18(i+2)(i^3 - 5ip^2 + 3i^2 - 7p^2 + 3i + 1)v_{i+1,j}^{(p)} \\ + (3i+4)(3i+5)(i+p+2)(i-p+2)v_{i+2,j}^{(p)} = 0\end{aligned}$$

and

$$\begin{aligned}\mathcal{O}_j v_{ij}^{(p)} = 9(j+1)^2(j+2)^2(j+3p)(j-3p)v_{ij}^{(p)} \\ + 18j(j+2)^2(j^3 - 5jp^2 + 3j^2 - 7p^2 + 3j + 1)v_{i,j+1}^{(p)} \\ + j(j+1)(3j+4)(3j+5)(j+p+2)(j-p+2)v_{i,j+2}^{(p)} = 0\end{aligned}$$

Note that these relations together with the requirements that $v^{(p)}_{p,3p} = 3^{6p}$, $v_{p,3p-1} = 4 \cdot 3^{6p-2}(3p-1)$ and similarly at the other corner are enough to uniquely specify the terms $v^{(p)}_{i,j}$ for every i, j, p. (Note that $v^{(p)}_{ij}$ is zero unless $p \le i, j \le 3p$.)

We'd like to deduce from these relations that $v_{ij} = \sum_p v^{(p)}_{ij}$ satisfies the recurrence relation (†) of §2 above; since it is easy to check that the initial conditions agree, it follows that $v_{ij} = u_{ij}$. But I see no obvious way of doing this. It is **not** a formal consequence of the two recurrence relations satisfied by $v^{(p)}_{ij}$, as the recurrence relation (†) does not hold for $v^{(p)}_{ij}$ for each individual p.

What is needed is a recurrence relation for $v^{(p)}_{ij}$ as a function of p. Unfortunately this appears to be computationally intractable, as the best upper bound one can find for the degree of the recurrence is about 15, which is way beyond what is feasible to compute.

References

[BC05] Kevin Buzzard and Frank Calegari, *Slopes of overconvergent 2-adic modular forms*, Compos. Math. **141** (2005), no. 3, 591–604. MR 2135279

[Loe07] David Loeffler, *Spectral expansions of overconvergent modular functions*, Int. Math. Res. Notices **2007** (2007), no. 050. MR 2353090

[PWZ96] Marko Petkovšek, Herbert S. Wilf, and Doron Zeilberger, $A = B$, A K Peters Ltd., 1996, With a foreword by Donald E. Knuth, With a separately available computer disk. MR 1379802

[Smi00] Lawren Smithline, *Slopes of p-adic modular forms*, Ph.D. thesis, Harvard University, 2000.

[Zei90] Doron Zeilberger, *A holonomic systems approach to special functions identities*, J. Comput. Appl. Math. **32** (1990), no. 3, 321–368. MR 1090884