A graph $G=(V,E)$ consists of two sets $V$ and $E$. The elements of $V$ are called vertices (nodes) and the elements of $E$ are called edges. Each edge is a pair of vertices.

**Example:** $V=\{1,2,3,4,5\}$, $E=\{\{1,2\},\{2,3\},\{3,4\},\{4,5\}\}$

By altering the definition, we can obtain different types of graphs:

- by replacing $E$ with a set of *ordered* pairs of vertices, we obtain an oriented (directed) graph

  ![Directed Graph Example](image)

- by allowing edges to connect a vertex to itself, we obtain a pseudograph
- by allowing $E$ to be a multiset, we obtain a multigraph
Basic Graph Terminology

A simple graph is a graph which is undirected, without loops and multiple edges.

- \(a\) and \(b\) are adjacent
- \(a\) and \(b\) are neighbors
- \(a\) and \(b\) are \(\in\) \(E(G)\)

The neighborhood \(N(v)\) of a vertex \(v\) is the set of vertices adjacent to \(v\).

The degree \(\deg(v)\) of a vertex \(v\) is the number of its neighbors, i.e., \(|N(v)|\).

Example: \(N(1) = \{2, 3, 6, 7\}\) \(\deg(1) = 4\)
Basic Graph Terminology

A simple graph is a graph which is undirected, without loops and multiple edges.

- \( a \) and \( b \) are adjacent
- \( a \) and \( b \) are neighbors
- \( ab \in E(G) \)

How many simple labeled graphs on \( n \) vertices are there?
Basic Graph Terminology

A simple graph is a graph which is undirected, without loops and multiple edges.

- a and b are adjacent
- a and b are neighbors
- \( ab \in E(G) \)

A graph \( G_1 = (V_1, E_1) \) is isomorphic to a graph \( G_2 = (V_2, E_2) \) if there is a bijection \( f: V_1 \rightarrow V_2 \) such that \( xy \in E_1 \) iff \( f(x)f(y) \in E_2 \)
Isomorphism
Isomorphism
Isomorphism
Exercise

How many pairwise non-isomorphic graphs on $n$ vertices are there?

How many pairwise non-isomorphic graphs on 3 vertices are there?
The complement of a graph

Let $G_1=(V,E_1)$ and $G_2=(V,E_2)$ be two graphs on the same vertex set $V$. $G_2$ is said to be the complement of $G_1$ if $ij \in E_1 \iff ij \notin E_2$

$G_1 = \overline{G_2}$
Exercise

How many pairwise non-isomorphic graphs on 4 vertices are there?
Exercise

Are there self-complementary graphs on 5 vertices?

Are there self-complementary graphs on 6 vertices?

Are there self-complementary graphs on 7 vertices?

Are there self-complementary graphs on 8 vertices?
Graphs

- $P_n$ path on $n$ vertices
- $C_n$ cycle with $n$ vertices
- $K_n$ complete graph on $n$ vertices
- $B_n$ hypercube $V(B_n)=\{0,1\}^n$

$a=(a_1,\ldots,a_n)$ is adjacent to $b=(b_1,\ldots,b_n)$ iff $a$ and $b$ differ in exactly one component.

The degree of vertex $v$, $\deg(v)$, is the number of its neighbors.
Handshake Lemma. Let $G=(V,E)$ be a graph with $m$ edges. Then

$$\sum_{v \in V} \deg(v) = 2m$$

Proof. Every edge connects 2 vertices. Therefore, in the sum $\sum_{v \in V} \deg(v)$ every edge is counted twice.

How many edges has a graph with the degree sequence (3,3,3,3,3)?

Corollary. In any graph, the number of vertices of odd degree is even.
Graphs

How many edges has a graph with the degree sequence (0,1,2,3)?

**Corollary.** *Every graph with at least two vertices has two vertices of the same degree.*
**Connectivity**

**Definition.** A graph is connected if any two vertices of the graph are joint by a path. Otherwise, it is disconnected.

-connected-

**Definition.** A maximal connected subgraph of G is called a connected component of G.

Exercise: show that the complement of a disconnected graph is connected

**Definition.** A maximal connected subgraph of G is called a connected component of G.
Connectivity

What is the maximum number of edges in a disconnected graph?

\[
\max_k \left( \binom{k}{2} + \binom{n-k}{2} \right) = \frac{(n-1)(n-2)}{2}
\]

What is the minimum number of edges in a connected graph?

\[
0 < k < n \quad \text{and} \quad n-k
\]
Independent sets and Cliques

An **independent set** in a graph is a subset of vertices no two of which are adjacent.

A **clique** is a subset of pairwise adjacent vertices.

The **independence number** is the size of a maximum independent set

The **clique number** is the size of a maximum clique

**Exercise.**

• What is the independence number and the clique number of $K_n$, $P_n$, $C_n$, $B_n$?

• Show that the vertices of $B_n$ can be partitioned into 2 independent sets.
**Bipartite graphs**

**Definition.** A graph $G$ is bipartite if $V(G)$ can be partitioned into two independent sets.

**Exercise.** Which of the following graphs are bipartite?

- $K_n, P_n, C_n$
- $K_{3,3}$
- Complete bipartite graph $K_{2,2}$

$C_4$ is a complete bipartite graph with parts of size $n$ and $m$.
Bipartite graphs

**Theorem.** A graph is bipartite if and only if it has no odd cycles.

**Proof.** Assume $G$ is a bipartite graph and let $C=(c_1, c_2, \ldots, c_k)$ be a cycle in $G$.

Therefore, $k$ is even.
**Theorem.** A graph is bipartite if and only if it has no odd cycles.

**Proof.** Assume $G$ is has no odd cycle. Without loss of generality $G$ is connected.

Let $P_x$ be a shortest path connecting $v$ to $x$, and $P_y$ be a shortest path connecting $v$ to $y$. Let $z$ be a common vertex of $P_x$ and $P_y$ closest to $x$ and $y$. Then $\text{dist}(z,x)$ and $\text{dist}(z,y)$ have the same parity. Therefore, $x$ is not adjacent to $y$, since otherwise an odd cycle arises.

$V_1 = \{ u \in V(G) : \text{dist}(u,v) \text{ is odd} \}$

Similarly, one can show that the set $V_2 = V(G) - V_1$ is also an independent set.
**Diameter**

**Distance** between two vertices $a$ and $b$ is the length of a shortest path connecting them.

**Diameter** of a connected graph $G$ is

$$\max_{a,b \in V(G)} \text{dist}(a,b)$$

What is the diameter of $C_n$, $P_n$, $K_n$, $B_n$?

**Exercise.** Find all bipartite graphs with diameter 2?
Diameter

**Theorem.** Diameter of almost all graphs is 2, i.e. \( \lim_{n \to \infty} |\Gamma^2(n)|/|\Gamma(n)| = 1. \)

- \( \Gamma(n) \) the set of all labeled graphs with \( n \) vertices
- \( \Gamma^2(n) \) the set of all labeled graphs of diameter 2 with \( n \) vertices

**Proof.** Since for any \( n \) there is exactly one graph of diameter 1 (\( K_n \)), diameter of almost all graphs is at least 2.

Let \( \Gamma'(n) \) be the set of all labeled graphs with \( n \) vertices of diameter more than 2. We will show that \( \lim_{n \to \infty} |\Gamma'(n)|/|\Gamma(n)| = 0. \)

Let \( G \) be a graph of diameter more than 2 and let \( u, v \) be two vertices of \( G \) of distance more than 2.
Let $\Gamma'_{u,v,U}(n)$ the set of graphs from $\Gamma'(n)$ with fixed vertices $u,v$ and fixed set $U$ as above. In each graph in $\Gamma'_{u,v,U}(n)$, $k$ pairs of vertices create edges and $n-1=k+(n-1-k)$ pairs create non-edges. Therefore,

$$\left| \Gamma'_{u,v,U}(n) \right| = 2^{\binom{n}{2} - k - n + 1}$$
Diameter

Let $\Gamma'_{u,U}(n) = \bigcup_v \Gamma'_{u,v,U}(n)$, then

$$| \Gamma'_{u,U}(n) | \leq (n - k - 1) | \Gamma'_{u,v,U}(n) | < n2^{\binom{n}{2} - k - n + 1}$$

Next, let $\Gamma'_{u}(n) = \bigcup_U \Gamma'_{u,U}(n)$, then

$$| \Gamma'_{u}(n) | \leq \sum_{k=0}^{n-2} \binom{n-1}{k} | \Gamma'_{u,U}(n) | < \sum_{k=0}^{n-2} \binom{n-1}{k} n2^{\binom{n}{2} - k - n + 1} = n2^{\binom{n}{2} - n + 1} \sum_{k=0}^{n-2} \binom{n-1}{k} 2^{-k}$$
By the Binomial Theorem, \((\frac{1}{2} + 1)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} 2^{-k}\), therefore,

\[\sum_{k=0}^{n-2} \binom{n-1}{k} 2^{-k} = (\frac{3}{2})^{n-1} - \frac{1}{2^{n-1}} < (\frac{3}{2})^{n-1}\]

Since \(|\Gamma'(n)| = n|\Gamma'_u(n)|\),

\[|\Gamma'_u(n)| < n2^{\binom{n}{2} - n + 1}(\frac{3}{2})^{n-1}\]

\[|\Gamma'(n)| < n^2 2^{\binom{n}{2} - n + 1}(\frac{3}{2})^{n-1}\]

\[\frac{|\Gamma'(n)|}{|\Gamma(n)|} < \frac{n^2 2^{\binom{n}{2} - n + 1}(\frac{3}{2})^{n-1}}{2^{\binom{n}{2}}} = \frac{n^2 (3/4)^{n-1}}{\binom{n}{2}} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty\]