Combinatorics

Vadim Lozin
What is Combinatorics?
Combinatorics is special. Most mathematical topics which can be covered in a lecture course build towards a single, well-defined goal, such as the Prime Number Theorem. Even if such a clear goal doesn’t exist, there is a sharp focus (e.g. finite groups). By contrast, combinatorics appears to be a collection of unrelated puzzles chosen at random.

Two factors contribute to this. First, combinatorics is broad rather than deep. Second, it is about techniques rather than results.

Combinatorics could be described as the art of arranging objects according to specified rules. We want to know, first, whether a particular arrangement is possible at all, and if so, in how many different ways it can be done.

Peter Cameron
School of Mathematical Sciences
Queen Mary, University of London
Sample Problems

Derangements

*Given n letters and n addressed envelopes, in how many ways can the letters be placed in the envelopes so that no letter is in the correct envelope?*
Kirkman’s schoolgirls

*Fifteen schoolgirls walk each day in five groups of three. Arrange the girls’ walks for a week so that, in that time, each pair of girls walks together in a group just once.*
<table>
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<tr>
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1847

_Fifteen schoolgirls walk each day in five groups of three. Arrange the girls’ walks for a week so that, in that time, each pair of girls walks together in a group just once._
Kirkman’s schoolgirls

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Thomas Penyngton Kirkman (31.03.1806–3.02.1895) was a British mathematician. An important expositor of group theory in English, he is now remembered principally for a combinatorial problem which bears his name, Kirkman's schoolgirl problem.

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More generally, it was shown in 1967 that the solution exists for any \( n \equiv 3 \) (modulo 6)

**Sample Problems**

**Kirkman’s schoolgirls**

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**Fifteen schoolgirls walk each day in five groups of three. Arrange the girls’ walks for a week so that, in that time, each pair of girls walks together in a group just once.**
Euler’s officers

Thirty-six officers are given, belonging to six regiments and holding six ranks (so that each combination of rank and regiment corresponds to just one officer). Can the officers be paraded in a $6 \times 6$ array so that in any line (row or column) of the array, each regiment and each rank occurs precisely once?
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Leonhard Paul Euler (15.04.1707 – 18.09.1783) was a pioneering Swiss mathematician and physicist who spent most of his life in Russia and Germany.

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n=2

(A,1), (A,2), (B,1), (B,2)

(A,1)(B,2)

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1960: solution exists for all n except n=2 and n=6

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Sample Problems

Ramsey Game

This two player game requires a sheet of paper and pencils of two colors, say red and blue. Six points on the paper are chosen, with no three in line. Now the players take a pencil each, and take turns drawing a line connecting two of the chosen points. The first player to complete a triangle of her own color loses. Can the game ever result in a draw?
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Ramsey Game

Frank Plumpton Ramsey (22.02.1903 – 19.01.1930) was a British mathematician who, in addition to mathematics, made significant and precocious contributions in philosophy and economics before his death at the age of 26.

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Object: move pile A to B by moving one disc at a time. A disc may never rest on a smaller one.
Sample Problems

Tower of Hanoi

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What is the minimum number of moves?
Sample Problems

Tower of Hanoi

Hanoi is the capital and second-largest city of Vietnam (estimated population 6 232 940).

n discs

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Chess tournament

A chess tournament has $n$ participants, and any two players play one game against each other. Is it true that in any given point of time, there are two players who have finished the same number of games?
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Are there graphs all of whose vertices have pairwise different degrees?

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Sample Problems

Timetabling

Consider a school in which there are m teachers $T_1, T_2, \ldots, T_m$ and n classes $C_1, C_2, \ldots, C_n$. The teacher $T_i$ has to teach the class $C_j$ a specified number $p_{i,j}$ of periods. What is the minimum possible number $d$ of periods in a complete timetable?
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$$d_i = p_{i,1} + p_{i,2} + \ldots + p_{i,n} = \sum_{j=1}^{n} p_{i,j} \quad \text{the total number of periods the teacher } T_i \text{ has to teach}$$
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By König’s Theorem, \( d = \max_{i,j} \{d_i, c_j\} \)
What is Combinatorics?

Combinatorics is a branch of pure mathematics concerning the study of discrete (and usually finite) objects. Aspects of combinatorics include

- "counting" the objects satisfying certain criteria (enumerative combinatorics),
- deciding when the criteria can be met,
- constructing and analyzing objects meeting the criteria (combinatorial designs),
- finding “largest” or “smallest” objects (extremal combinatorics),
- finding algebraic structures these objects may have (algebraic combinatorics).

One of the oldest and most accessible parts of combinatorics is graph theory, which also has numerous natural connections to other areas.

Wikipedia
Mathematical Subject Classification

05
Combinatorics

05A
Enumerative combinatorics

05B
Designs and configurations

05C
Graph theory

05D
Extremal combinatorics

05E
Algebraic combinatorics
Basic Course Outline

I - Enumerative combinatorics

• Basic counting
• Combinatorial identities
• Combinatorial functions
• Combinatorial numbers

II - Graph Theory

• Basic concepts
• Hereditary properties
• Trees
• Planarity
• Matching Theory
References

http://www.warwick.ac.uk/~masgax/MA241.htm

http://www2.warwick.ac.uk/fac/sci/maths/undergrad/ughandbook/year2/ma241/
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Lecture Notes:
Enumerative Combinatorics:
http://www.maths.warwick.ac.uk/~daan/MA241Combinatorics0607/Combin06.pdf
Graph Theory:
http://www.warwick.ac.uk/~masgax/notes.pdf
References

http://www.warwick.ac.uk/~masgax/MA241.htm
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Graph Theory:
http://www.warwick.ac.uk/~masgax/notes.pdf

Recommendable books:
• *Foundations of Combinatorics with Applications*, Edward E. Bender and S. Gill Williamson, Dover Publications 2006 (available online at the authors' webpage)
• *Combinatorics and Graph Theory*, J.M. Harris, J.L. Hirst and M.J. Mossinghoff, Springer, 2000 (scanned copy of Chapter 1 is available at http://www2.warwick.ac.uk/services/library/main/electronicresources/extracts/ma/ma241 )