Stirling numbers of the first kind

**Definition.** The Stirling number of the first kind \( [n \atop k] = s(n, k) \) is the number of permutations of \( n \) elements consisting of \( k \) cycles.

Example: \( n=4, k=2 \)

\[
(1,2,3)(4), (1,3,2)(4), (1,2,4)(3), (1,4,2)(3), (1,3,4)(2), (1,4,3)(2), \\
(2,3,4)(1), (2,4,3)(1), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)
\]

\[
\begin{align*}
[n \atop k] &= 0 \quad \text{if} \quad k < 0 \quad \text{or} \quad k > n \\
[n \atop 0] &= 1, \quad n = 0 \\
[n \atop n] &= 1
\end{align*}
\]

\[
\begin{align*}
[n \atop 1] &= (n - 1)! \\
[n \atop n-1] &= \binom{n}{2} \\
\sum_{k \geq 0} [n \atop k] &= n!
\end{align*}
\]
Stirling numbers of the first kind

Basic recurrence relation

\[
\begin{bmatrix} n \\ k \end{bmatrix} = (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}
\]

**Proof.** We divide all permutations with k cycles into two groups

A: permutations containing (n) as a cycle,
B: all remaining permutations.

Any permutation in A can be obtained from a permutation of n-1 elements with k-1 cycles (there are s(n-1,k-1) such permutations) by adding to it (n).

Any permutation in B can be obtained from a permutation of n-1 elements with k cycles (s(n-1,k) permutations) by inserting the element n into one of the cycles (n-1 ways).
Properties of falling and rising powers

\[ x^k = x(x-1)...(x-k+1) \]  \hspace{1cm} \text{Falling power}

\[ x^\overline{k} = x(x+1)...(x+k-1) \]  \hspace{1cm} \text{Rising power}

\[ x^0 = 1 \]
\[ (x + 1)^m - x^m = mx^{m-1} \]
\[ x^0 = 1 \]
\[ x^m - (x-1)^m = mx^{m-1} \]
\[ n^n = n! \]

\[ 0^{-k} = \frac{1}{(-1)(-2)\cdots(-k)} \]
\[ x^{-k} = \frac{1}{(x+1)^{-k}} \]
\[ 0^{-k} = \frac{1}{1 \cdot 2 \cdot 3 \cdots k} \]
\[ x^{-\overline{k}} = \frac{1}{(x-1)^{-k}} \]

\[ x^3 = x(x-1)(x-2) \quad \text{/}(x-2) \]
\[ x^2 = x(x-1) \quad \text{/}(x-1) \]
\[ x^1 = x \quad \text{/}x \]
\[ x^0 = 1 \quad \text{/}(x+1) \]
\[ x^{-1} = \frac{1}{x + 1} \quad \text{/}(x+2) \]
\[ x^{-2} = \frac{1}{(x + 1)(x + 2)} \]
Properties of Stirling numbers

Theorem

\[ x^n = \sum_k \left\{ \begin{array}{c} n \\ k \end{array} \right\} x^k \]

\[ x^n = \sum_k (-1)^{n-k} \left\{ \begin{array}{c} n \\ k \end{array} \right\} x^k \]

Proof.

\[ x^n = x \cdot x^{n-1} = x \sum_k \left\{ \begin{array}{c} n-1 \\ k \end{array} \right\} x^k \]

\[ = \sum_k \left\{ \begin{array}{c} n-1 \\ k \end{array} \right\} (x^{k+1} + kx^k) \]

\[ = \sum_k \left\{ \begin{array}{c} n-1 \\ k \end{array} \right\} x^{k+1} + \sum_k k \left\{ \begin{array}{c} n-1 \\ k \end{array} \right\} x^k \]

\[ = \sum_k \left\{ \begin{array}{c} n-1 \\ k-1 \end{array} \right\} x^k + \sum_k k \left\{ \begin{array}{c} n-1 \\ k \end{array} \right\} x^k \]

\[ = \sum_k \left\{ \begin{array}{c} n \\ k \end{array} \right\} x^k \]

Can be proved by analogy
Properties of Stirling numbers

Theorem

\[ x^n = \sum_{k} \left[ \begin{array}{c} n \\ k \end{array} \right] x^k \]

\[ x^n = x^{n-1} (x+n-1) = (x+n-1) \sum_{k} \left[ \begin{array}{c} n-1 \\ k \end{array} \right] x^k \]

\[ = x \sum_{k} \left[ \begin{array}{c} n-1 \\ k \end{array} \right] x^k + \sum_{k} (n-1) \left[ \begin{array}{c} n-1 \\ k \end{array} \right] x^k \]

\[ = \sum_{k} \left[ \begin{array}{c} n-1 \\ k \end{array} \right] x^{k+1} + \sum_{k} (n-1) \left[ \begin{array}{c} n-1 \\ k \end{array} \right] x^k \]

\[ = \sum_{k} \left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right] x^k + \sum_{k} (n-1) \left[ \begin{array}{c} n-1 \\ k \end{array} \right] x^k \]

\[ = \sum_{k} \left[ \begin{array}{c} n \\ k \end{array} \right] x^k \]

Can be proved by analogy
The Bell number $B_n$ is the number of partitions of an $n$-set.

$$B_n = \sum_{k=1}^{n} \binom{n}{k}$$

Example: $n=3$

$$\{1,2,3\}, \{1\} \cup \{2\} \cup \{3\}, \{1,2\} \cup \{3\}, \{1,3\} \cup \{2\}, \{2,3\} \cup \{1\}$$

Recurrence relation:

$$B_n = \sum_{k=1}^{n} \binom{n-1}{k-1} B_{n-k}$$

Proof: Let $X=\{1,2,\ldots,n\}$ and let $X_1 \cup X_2 \cup \ldots \cup X_p$ be a partition of $X$. Assume $n \in X_p$, i.e., $X_p = \{n\} \cup Y$, where $Y \subseteq \{1,2,\ldots,n-1\}$. Denote $k = |X_p|$. There are $C(n-1,k-1)$ ways to choose the set $Y$ and there are $B_{n-k}$ ways to partition the elements of $X-X_p$. Summing over all possible values of $k=1,2,\ldots,n$ gives the result.

$$B_0 = 1$$
Frank Plumpton Ramsey (February 22, 1903 – January 19, 1930) was a British mathematician who, in addition to mathematics, made significant contributions in philosophy and economics.
Ramsey Theory

Pigeonhole Principle

If $n+1$ letters are placed in $n$ pigeonholes, then some pigeonhole must contain more than one letter.

Let $r$ and $p$ be positive integers. Then there is an $n$ such that for any coloring of $n$ objects with $r$ different colors there exist $p$ objects of the same color.

$$n \geq r(p-1)+1$$
Ramsey Theory

Let \( r, k, p \) be positive integers. Then there is a positive integer \( n \) with the following property. If the \( k \)-subsets of an \( n \)-set are colored with \( r \) colors, then there is a monochromatic \( p \)-set, i.e., a \( p \)-set all of whose \( k \)-subsets have the same color.

\( r=2, \ k=2 \)

For each positive integer \( p \), there is a positive integer \( n \) such that every coloring of the edges of the complete graph \( K_n \) with 2 colors contains a complete monochromatic subgraph on \( p \) vertices.
Ramsey Theory

Let $r, k, p$ be positive integers. Then there is a positive integer $n$ with the following property. If the $k$-subsets of an $n$-set are colored with $r$ colors, then there is a monochromatic $p$-set, i.e., a $p$-set all of whose $k$-subsets have the same color.

$r=2, k=2$

For each positive integer $p$, there is a positive integer $n$ such that every coloring of the edges of the complete graph $K_n$ with 2 colors contains a complete monochromatic subgraph on $p$ vertices.

For each positive integer $p$, there is a positive integer $n$ such that every graph with $n$ vertices contains either a clique of size $p$ or an independent set of size $p$. 
Ramsey Theory

Ramsey Numbers

The minimum $n$ such that every graph with $n$ vertices contains either a clique of size $p$ or an independent set of size $p$ is the symmetric Ramsey number $R(p)$.

The minimum $n$ such that every graph with $n$ vertices contains either a clique of size $p$ or an independent set of size $q$ is the Ramsey number $R(p,q)$.

$R(p) = R(p,p)$

$R(p,q) = R(q,p)$
**Theorem.** $R(3)=6$

**Proof.** $R(3)\leq 6$, i.e., every graph with at least 6 vertices contains either a triangle (clique of size 3) or its complement (independent set of size 3).

Let $G$ be a graph with 6 vertices and $v$ a vertex of $G$. By the Pigeonhole principle, $v$ is incident to at least 3 edges or 3 non-edges. Assume w.l.o.g. $v$ has three neighbors. If at least two of its neighbors are adjacent, then $G$ has a triangle, otherwise the neighbors of $v$ form an anti-triangle.

$R(3)\geq 6$, because $C_5$ contains neither triangle nor its complement.
Ramsey Theory

Ramsey Numbers

*Exercise.* Determine $R(2) = 2$, $R(2,3) = 3$, $R(3,2) = 3$

Determine $R(2, k) = k = R(k, 2)$

*Theorem.* $R(3, 4) = 9$

*Proof.* $R(3, 4) \geq 9$

no triangles

no independents sets of size 4
Ramsey Theory

Theorem. \( R(3,4)=9 \)

Proof. \( R(3,4) \leq 9 \) Let \( G \) be a graph with 9 vertices and \( v \) a vertex of \( G \).

1. Assume \( v \) has 4 neighbors. Then \( G \) has either a triangle or an independent set of size 4.

2. Assume \( v \) has 6 non-neighbors. The subgraph induced by these 6 non-neighbors contains either a triangle, in which case \( G \) also has a triangle, or an anti-triangle, in which case this anti-triangle together with \( v \) form an independent set of size 4.

3. If \( v \) has less than 4 neighbors and less than 6 non-neighbors, then the degree of \( v \) must be exactly 3. Since \( v \) has been chosen arbitrarily, the degree of each vertex of \( G \) is 3, which is impossible by the Handshake Lemma.
**Theorem.** \( R(4,4) = 18 \)

**Proof.** \( R(4,4) \leq 18 \). Let \( G \) be a graph with 18 vertices and \( v \) a vertex of \( G \).

By the Pigeon principle, \( v \) has either 9 neighbors or 9 non-neighbors. Since \( R(4,4) \) is symmetric, we let w.l.o.g. \( v \) have 9 neighbors. Since \( R(3,4) = 9 \), the 9 neighbors of \( v \) induce a subgraph which contains

- either an independent set of size 4, in which case \( G \) also has an independent set of size 4,
- or a triangle, which together with \( v \) form a clique of size 4.

\( R(4,4) \geq 18 \). Exercise
Ramsey Theory

Some other Ramsey Numbers

\[ R(2,k) = k \]
\[ R(3,3) = 6, \; R(3,4) = 9, \; R(3,5) = 14, \; R(3,6) = 18, \; R(3,7) = 23, \; R(3,8) = 28, \; R(3,9) = 36 \]
\[ R(4,4) = 18, \; R(4,5) = 25 \]

Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of \( R(5,5) \) or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for \( R(6,6) \). In that case, he believes, we should attempt to destroy the aliens before they destroy us.
**Theorem.** If $p>1$ and $q>1$, then $R(p,q) \leq R(p-1,q)+R(p,q-1)$

**Proof.** Let $G$ be a graph with $R(p-1,q)+R(p,q-1)$ vertices and $v$ a vertex of $G$. Denote by $G_1$ the subgraph of $G$ induced by the neighbors of $v$, and by $G_2$ the subgraph induced by the non-neighbors of $v$.

Since $|V(G_1)|+|V(G_2)|+1=R(p-1,q)+R(p,q-1)$, we have either $|V(G_1)| \geq R(p-1,q)$ or $|V(G_2)| \geq R(p,q-1)$.

If $|V(G_1)| \geq R(p-1,q)$, $G_1$ contains either an independent set of size $q$, in which case we are done, or a clique of size $p-1$. This clique together with $v$ create a clique of size $p$, and we are done again. The case $|V(G_2)| \geq R(p,q-1)$ is similar.

**Corollary.** $R(4,4) \leq R(3,4)+R(4,3)=18$
Ramsey Theory

Bounds on the Ramsey number

**Theorem.** If $p,q \geq 2$, then $R(p,q) \leq \binom{p+q-2}{p-1}$

**Proof.** Induction on $p$ and $q$. For $p=2$ we have $R(2,q) = q = \binom{2+q-2}{2-1}$ and similarly for $q=2$.

We know that $R(p,q) \leq R(p-1,q) + R(p,q-1)$ and therefore by inductive hypothesis we have

$$R(p,q) \leq R(p-1,q) + R(p,q-1) \leq \binom{p-1+q-2}{p-2} + \binom{p+q-3}{p-1} = \binom{p+q-2}{p-1}$$

Pascal Triangle
Ramsey Theory

Bounds on the Ramsey number

Theorem. $R(p, p) \geq 2^{(p-2)/2}$

There are $2^{n(n-1)/2}$ graphs with $n$ vertices. The proportion of graphs that have a monochromatic set of size $p$ (a clique or independent set of size $p$) does not exceed

$$\frac{\binom{n}{p} \cdot 2^{\binom{n}{2} - \binom{p}{2}}}{2^{\binom{n}{2}}} = \left(\frac{n}{p}\right)^{2^{1-\binom{p}{2}}}$$

For $n = \left\lfloor 2^{(p-2)/2} \right\rfloor$ we have

$$\left(\frac{n}{p}\right)^{2^{1-p(p-1)/2}} < n^p 2^{-p(p-2)/2} \leq 1$$

because

$$\left(\frac{n}{p}\right) < n^p \quad \text{and} \quad 1 - p(p-1)/2 < -p(p-2)/2$$

Since the proportion is strictly less than 1, there are graphs with $n = \left\lfloor 2^{(p-2)/2} \right\rfloor$ vertices that do not have monochromatic sets of size $p$. Therefore,

$R(p, p) > n = \left\lfloor 2^{(p-2)/2} \right\rfloor$
Definition. A class of graphs $X$ is hereditary if $G \in X$ implies $G-v \in X$ for $\forall v \in V(G)$ (if with any graph $G$ it contains all induced subgraphs of $G$)

Examples: bipartite graphs = Free($C_3,C_5,C_7,...$) 
permutation graphs $\subset$ Free($C_5$) 
comparability graphs $\subset$ Free($C_5$) 
planar graphs $\subset$ Free($K_5,K_3,3$) 
complete graphs = Free($O_2$), where $O_2$ is the complement of $K_2$ 
line graphs $\subset$ Free($K_1,3$) 
tress $\subset$ forests = graphs without cycles = Free($C_3,C_4,C_5,...$)

For a set of graphs $M$, let Free($M$) denote the class of graphs containing no induced subgraphs from $M$. The graphs in $M$ are called forbidden induced subgraphs.

Theorem. A class of graphs $X$ is hereditary if and only if $X=\text{Free}(M)$ for some set $M$.

Proof. If $X$ is hereditary, then $X=\text{Free}(M)$ with $M$ being the set of all graphs not in $X$. If $X=\text{Free}(M)$ and $G \in X$, then obviously $G-v \in \text{Free}(M)=X$. 

Hereditary classes of graphs
Hereditary classes of graphs defined by small forbidden induced subgraphs

Free($P_3$) = the class of graphs every connected component of which is a clique

Obviously, Free($P_3$) $\subseteq$ the class of graphs every connected component of which is a clique, because $P_3$ is a connected graph different from a clique.

To see that Free($P_3$) $\subseteq$ the class of graphs every connected component of which is a clique, consider a graph $G \in$ Free($P_3$) and assume by contradiction that $G$ contains a connected component which is not a clique. Then this component has two non-adjacent vertices. Any path connecting these vertices has length at least two, which means $G$ contains a $P_3$. A contradiction.

The number of n-vertex labeled graphs in Free($P_3$)? Bell number $B_n$

Free($K_2$) = the class of complete graphs
The number of n-vertex graphs is 1

Free($O_2$) = the class of complete graphs
Free($K_2$) = the class of empty (edgeless) graphs
Free($O_3$) = the class of empty (edgeless) graphs

Free($P_3$) = the class of graphs every connected component of which is a clique

Free($K_2 + K_1$) = the graphs complement to graphs in Free($P_3$) (complete k-partite graphs)

Free($K_3$) $\Rightarrow$ bipartite graphs
The number of n-vertex labeled graphs $\geq 2^{\frac{n^2}{4}}$
Hereditary classes of graphs defined by small forbidden induced subgraphs

\[ \text{Free}(K_3, P_3) = \text{Free}(K_3) \cap \text{Free}(P_3) = \text{the class of graphs every connected component of which is a clique of size at most 2} \]

The number of \( n \)-vertex labeled graphs = The number of involutions on \( n \) elements

\[ \text{Free}(K_3, O_3) \quad \text{The number of } n \text{-vertex labeled graphs} = 0 \text{ for } n \geq 6 \]

**Theorem.** The class \( \text{Free}(M) \) is finite if and only if \( M \) contains both a complete graph and an empty graph.

**Proof.** If \( M \) contains both a complete graph \( K_n \) and an empty graph \( O_m \), then \( \text{Free}(M) \) contains no graphs with at least \( R(n,m) \) vertices. If \( M \) does not contain any complete (empty) graph, then \( \text{Free}(M) \) contains all complete (empty) graphs, of which there are infinitely many.

**Theorem.** \( \text{Free}(M) \cap \text{Free}(N) = \text{Free}(N \cup M) \)
Let \( K \) be the class of all complete graphs and \( O \) be the class of all empty (edgeless) graphs.

**Theorem.** A hereditary class \( X \) is finite if and only if neither \( K \) nor \( O \) is a subclasses of \( X \).

**Definition.** An infinite hereditary class \( X \) is *constant* if there is a constant \( N \) such that for every \( n \geq 1 \) the number of \( n \)-vertex graphs in \( X \) is bounded by \( N \).

Let \( S \) be the class of stars, i.e. graphs of the form \( K_{1,p} \) and all their induced subgraphs. \( E^1 \) be the class of graphs with at most one edge.

For any hereditary class \( X \), denote by

- \( X_n \) the set of \( n \)-vertex graphs in \( X \)
- \( X^{\text{comp}} \) the class of complements of graphs in \( X \).

For any hereditary class \( X \), \(|X_n| = |X^{\text{comp}}|\).

\[ |S_n| = n + 1 \]
\[ |E^1_n| = n(n-1)/2 + 1 \]
Theorem. An infinite hereditary class \( X \) is constant if and only if none of the following classes is a subclass of \( X \): \( \mathbf{S}, \mathbf{E}^1, \mathbf{S}^{\text{comp}}, \mathbf{E}^{1\text{comp}}. \)

Proof. None of the four classes is constant. Therefore if \( X \) contains at least one of them, then \( X \) is not constant.

Conversely, assume \( X \) contains none of the four classes. In order to prove that \( X \) is constant, we will show that \( Y := X - (\mathbf{K} \cup \mathbf{O}) \) is a finite set. Suppose by contradiction that \( Y \) contains infinitely many graphs. Then by Ramsey's Theorem in graphs in \( Y \) there are either arbitrarily large independent sets or arbitrarily large cliques. Suppose the first is true, i.e. for any \( k \), there is a graph \( G \) in \( Y \) with an independent set of size \( k \). Since \( G \) contains at least one edge, it must contain an induced subgraph of the form \( K_{1,s} + O_{k-s} \) for some \( s \geq 1 \). Since \( G \) belongs to \( Y \), it also belongs to \( X \), and therefore, all induced subgraphs of \( G \) belong to \( X \) (because \( X \) is hereditary). In particular, all graphs of the form \( K_{1,s} + O_{k-s} \) for some \( s \geq 1 \) belong to \( X \), and since none of these graphs is empty or complete, all of them belong to \( Y \). Among these graphs there must exist either graphs with arbitrarily large value of \( k-s \) (in which case \( \mathbf{E}^1 \subseteq X \)) or with arbitrarily large value of \( s \) (in which case \( \mathbf{S} \subseteq X \)), since \( k \) can be arbitrarily large. A contradiction.

Similarly, a contradiction arises if graphs in \( Y \) contain arbitrarily large cliques.