Counting unlabeled full binary rooted plane trees

How many ways are there to place \( k-2 \) pairs of parentheses in the product \( x_1 \cdot \ldots \cdot x_k \) in such a way that the order of multiplication is completely specified?

\[ k=3 \quad (x_1x_2)x_3 \quad x_1(x_2x_3) \]

\[ k=4 \quad ((x_1x_2)x_3)x_4 \quad (x_1(x_2x_3))x_4 \quad (x_1x_2)(x_3x_4) \quad x_1((x_2x_3)x_4) \quad x_1(x_2(x_3x_4)) \]
Counting unlabeled full binary rooted plane trees

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$k=4 \quad ((x_1x_2)x_3)x_4 \quad (x_1(x_2x_3))x_4 \quad (x_1x_2)(x_3x_4) \quad x_1((x_2x_3)x_4) \quad x_1(x_2(x_3x_4))$
Counting unlabeled full binary rooted plane trees

How many ways are there to cut a regular n-gon into triangles?

$n=4$

$n=5$

Lemma. For $n>3$, any triangulation of an n-gon has $n-3$ diagonals, $n-2$ triangles, and at least 2 triangles containing two edges of the original n-gon.

Proof. Let $D$ be the number of diagonals, $T$ the number of triangles, and $T_i$ the number of triangles containing $i$ edges of the n-gon. Then $T_0+T_1+T_2=T$, $T_1+2T_2=n$, $3T=n+2D$. Let’s show that $D=n-3$ by induction on $n$. Cut the n-gon along any diagonal into two polygons with $k_1$ and $k_2$ sides. Then $k_1+k_2=n+2$, $D=(k_1-3)+(k_2-3)+1=n-3$. 
Counting unlabeled full binary rooted plane trees

How many ways are there to cut a regular n-gon into triangles?

n=4

1 2 3 4

n=5

1 2 3 4 5

(12)
Counting unlabeled full binary rooted plane trees

How many ways are there to cut a regular n-gon into triangles?

n=4

n=5

(12)(34)
Counting unlabeled full binary rooted plane trees

How many ways are there to cut a regular n-gon into triangles?

\[
\begin{align*}
n=4 & \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \quad 2 \quad 3 \\
4
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \quad 2 \quad 3 \\
4
\end{array}
\end{array}
\end{array} \\
(12)(34) & \quad (23)
\end{align*}
\]

\[
\begin{align*}
n=5 & \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \quad 2 \quad 3 \quad 4 \quad 5 \\
1 \quad 5
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
23
\end{array}
\end{array}
\end{array} \\
(23)
\end{align*}
\]
Counting unlabeled full binary rooted plane trees

How many ways are there to cut a regular n-gon into triangles?

n=4

n=5
Counting unlabeled full binary rooted plane trees

How many ways are there to cut a regular \( n \)-gon into triangles?

\( n=4 \)

\[ \begin{array}{c}
\text{1} & \text{4} \\
\text{2} & \text{3} \\
\end{array} \]

\( n=5 \)

\[ \begin{array}{c}
\text{1} & \text{5} \\
\text{2} & \text{4} \\
\text{3} & \text{4} \\
\text{4} & \text{5} \\
\text{5} & \text{5} \\
\end{array} \]

(12)(34)

(1(23))4
Counting unlabeled full binary rooted plane trees

How many ways are there to cut a regular n-gon into triangles?

n=4

n=5

(12)(34)  (1(23))4  1(2(34))  ((12)3)4  1((23)4)
Counting unlabeled full binary rooted plane trees

How many ways are there to cut a regular n-gon into triangles?

n=4

n=5

(12)(34)  (1(23))4  1(2(34))  ((12)3)4  1((23)4)
Counting unlabeled full binary rooted plane trees

**Definition.**
1. A single vertex is a UFBRP tree.
2. If $T_1$ and $T_2$ are UFBRP trees, then

\[
\begin{array}{c}
\text{is a UFBRP tree}
\end{array}
\]

$T_1 \quad \quad T_2$
Counting unlabeled full binary rooted plane trees

\[ b_n \] is the number of unlabeled full binary rooted plane trees with \( n \) leaves (i.e., with \( n \) non-parent vertices)

- \( b_0 = 0 \)
- \( b_1 = 1 \) (a single vertex is a leaf)
- \( b_2 = 1 \)
- \( b_3 = 2 \)
- \( b_4 = 5 \)
- \( b_5 = b_1b_4 + b_2b_3 + b_3b_2 + b_4b_1 = 14 \)

\[ b_n = \sum_{k=1}^{n-1} b_k b_{n-k} \]

Recursion
Counting unlabeled full binary rooted plane trees

$b_n$ is the number of unlabeled full binary rooted plane trees with $n$ leaves (i.e., with $n$ non-parent vertices)

$b_0=0$

$b_1=1$ (a single vertex is a leaf)

$b_2=1$

$b_3=2$

$b_4=5$

$b_5=b_1b_4+b_2b_3+b_3b_2+b_4b_1=14$

$$b_n = \sum_{k=0}^{n} b_k b_{n-k}$$

recursion

$n>1$
Counting unlabeled full binary rooted plane trees

\( b_n \) is the number of unlabeled full binary rooted plane trees with \( n \) leaves (i.e., with \( n \) non-parent vertices)

\( b_0 = 0 \)

\( b_1 = 1 \) (a single vertex is a leaf)

\( b_2 = 1 \)

\( b_3 = 2 \)

\( b_4 = 5 \)

\( b_5 = b_1 b_4 + b_2 b_3 + b_3 b_2 + b_4 b_1 = 14 \)

\[ b_n = \sum_{k=0}^{n} b_k b_{n-k} + a_n \quad \text{for } n \geq 0 \]

recursion \( a_1 = 1 \) and for \( n \neq 1 \), \( a_n = 0 \)
Generating functions

**Definition.** Let $a_0,a_1,a_2,...$ be an arbitrary sequence of numbers. The generating function for this sequence (associated with the sequence) is the function $G(x)$ whose value at $x$ is $\sum_{i\geq 0} a_i x^i$ (power series). The sequence $a_0,a_1,...$ is called the coefficients of the generating function.

**Examples.**

$$(1 + x)^n = \sum_{k \geq 0} \binom{n}{k} x^k$$

i.e., $(1+x)^n$ is the generating function for the binomial coefficients.
Generating functions

If $A(x)$ and $B(x)$ are two generating functions with coefficients $\{a_n\}$ and $\{b_n\}$, then the product $A(x)B(x)$ is a generating function with coefficients $\{c_n\}$ defined as follows:

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{i+j=n} a_i b_j$$

Convolution

$$A(x)B(x) = \left( \sum_{i \geq 0} a_i x^i \right) \left( \sum_{j \geq 0} b_j x^j \right) = \sum_{i,j \geq 0} a_i b_j x^{i+j} = \sum_{n \geq 0} \sum_{i+j=n} a_i b_j x^n$$
Counting unlabeled full binary rooted plane trees

\[ b_n = \sum_{k=0}^{n} b_k b_{n-k} + a_n \quad \text{a}_1=1 \text{ and for } n\neq 1, \text{a}_n=0 \]

\[ B(x) = \sum_{n\geq 0} b_n x^n \quad \text{Generating function for } \{b_n\} \]

\[ B(x) = \sum_{n\geq 0} \sum_{k=0}^{n} b_k b_{n-k} x^n + x \]

\[ B(x) = B(x) B(x) + x \quad B(x) = \frac{(1 \pm \sqrt{1-4x})}{2} \]

Since \(B(0)=0\), we conclude that

\[ B(x) = \frac{1 - \sqrt{1-4x}}{2} \]
Counting unlabeled full binary rooted plane trees

\[ B(x) = \frac{1}{2} - \frac{(1 - 4x)^2}{2} = \sum_{n \geq 0} b_n x^n \]

\[ (1 - 4x)^2 = \sum_{n \geq 0} \binom{1}{2} \binom{n}{2} (-4x)^n \quad \text{by Binomial Theorem} \]

\[ B(x) = \frac{1}{2} + \sum_{n \geq 0} (-\frac{1}{2}) \binom{1}{2} \binom{n}{2} (-4)^n x^n \]

\[ b_n = \left(-\frac{1}{2}\right) \binom{1}{2} (-4)^n \quad \text{n>0} \quad b_0 = \left(-\frac{1}{2}\right) \binom{1}{2} (-4)^0 + \frac{1}{2} = 0 \]
\[
\binom{n}{k} = n^k = \frac{n(n-1)...(n-k+1)}{k!}
\]

**Counting unlabeled full binary rooted plane trees**

\[
b_n = \left( -\frac{1}{2} \right)^n \binom{1}{n} (-4)^n = 2^{n-1} \frac{1}{2} \frac{1}{2} \cdots \frac{1}{2} - n + 1 \frac{1}{n!} 2(-2)^{n-1}
\]

\[
= 2^{n-1} (n-1)! \frac{(-1+2)(-1+4)\cdots(-1+2n-2)}{(n-1)!} \frac{1}{n!}
\]

\[
= \frac{2 \cdot 4 \cdots (2n-2)}{(n-1)!} \frac{1 \cdot 3 \cdots (2n-3)}{n!}
\]

Catalan numbers

\[
C_n = \frac{1}{n+1} \binom{2n}{n}
\]
**Generating functions**

**Definition.** Let \( a_0, a_1, a_2, \ldots \) be an arbitrary sequence of numbers. The generating function for this sequence (associated with the sequence) is the function \( G(x) \) whose value at \( x \) is \( \sum_{i \geq 0} a_i x^i \). The sequence \( a_0, a_1, \ldots \) is called the coefficients of the generating function.

**Examples.**

\[
(1 + x)^n = \sum_{k \geq 0} \binom{n}{k} x^k \quad \text{i.e., } (1+x)^n \text{ is the generating function for the binomial coefficients}
\]

\[
(1 - x)^{-n} = \sum_{k \geq 0} \binom{n+k-1}{k} x^k \Rightarrow \frac{a}{1 - rx} = \sum_{k \geq 0} ar^k x^k
\]
The number of binary words with no consecutive 1’s

Examples. 00100010 01010101

Denote by $F_n$ the number of words of this type of length $n$

- $F_1=2$  $F_2=3$  $\Rightarrow$  $F_0=1$ (there is one word of length 0)

- $a_1,a_2,...,a_{n-1},a_n$  $n>2$

If $a_n=0$, then $a_1,a_2,...,a_{n-1}$ is an arbitrary word of this type $F_{n-1}$

If $a_n=1$, then $a_{n-1}=0$ and $a_1,...,a_{n-2}$ is an arbitrary word of this type $F_{n-2}$

$F_n=F_{n-1}+F_{n-2}$  Fibonacci numbers

An alternative definition:  $F_1=1$,  $F_2=1$,  $F_n=F_{n-1}+F_{n-2}$  $\Rightarrow$  $F_0=0$
Generating function for the Fibonacci sequence

\[ G(x) = \sum_{n \geq 0} F_n x^n \]
Generating function for the Fibonacci sequence

\[ G(x) = \sum_{n \geq 0} F_n x^n = F_0 + F_1 x + \sum_{n \geq 2} F_n x^n \]

\[ = x + \sum_{n \geq 2} (F_{n-1} + F_{n-2}) x^n \]

\[ = x + x \sum_{n \geq 2} F_{n-1} x^{n-1} + x^2 \sum_{n \geq 2} F_{n-2} x^{n-2} \]

\[ = x + x \sum_{n \geq 1} F_n x^n + x^2 \sum_{n \geq 0} F_n x^n \]

\[ = x + x G(x) + x^2 G(x) \]

\[ G(x) = \frac{-x}{x^2 + x - 1} \]
Generating function for the Fibonacci sequence

\[ \frac{a}{1-rx} = \sum_{k \geq 0} ar^k x^k \]

\[ \left\{ \begin{array}{l}
A + B = -1 \\
A \phi^* + B \phi = 0
\end{array} \right. \]

\[ G(x) = \frac{-x}{x^2 + x - 1} = \frac{-x}{(x + \phi)(x + \phi^*)} = \frac{A}{x + \phi} + \frac{B}{x + \phi^*} = \frac{A(x + \phi^*) + B(x + \phi)}{(x + \phi)(x + \phi^*)} \]

$x^2 + x - 1 = 0$ \hspace{1cm} Golden ratio \hspace{1cm} $\phi = \frac{1 + \sqrt{5}}{2}$

$x^2 + x - 1 = (x + \phi)(x + \phi^*)$ \hspace{1cm} $\phi^* = \frac{1 - \sqrt{5}}{2}$

$\phi + \phi^* = 1$

$\phi\phi^* = -1$

$\phi - \phi^* = \sqrt{5}$
\[
\frac{a}{1-rx} = \sum_{k=0}^{\infty} ar^k x^k 
\]

**Generating function for the Fibonacci sequence**

\[
G(x) = \frac{-x}{x^2 + x - 1} = \frac{-x}{(x+\phi)(x+\phi^*)} = \frac{A}{x+\phi} + \frac{B}{x+\phi^*} = \frac{A(x+\phi^*)+B(x+\phi)}{(x+\phi)(x+\phi^*)}
\]

\[
x^2 + x - 1 = 0 \quad \text{Golden ratio} \quad \phi = \frac{1 + \sqrt{5}}{2}
\]

\[
x^2 + x - 1 = (x+\phi)(x+\phi^*) \quad \phi^* = \frac{1 - \sqrt{5}}{2}
\]

\[
A = -\frac{\phi}{\sqrt{5}} \quad B = \frac{\phi^*}{\sqrt{5}} \quad \phi + \phi^* = 1
\]

\[
\phi\phi^* = -1 
\]

\[
\phi - \phi^* = \sqrt{5}
\]

\[
G(x) = \frac{1}{\sqrt{5}} \left( \frac{\phi^*}{x+\phi^*} - \frac{\phi}{x+\phi} \right) = \frac{1}{\sqrt{5}} \left( \frac{1}{1-\phi x} - \frac{1}{1-\phi^* x} \right)
\]
Generating function for the Fibonacci sequence

\[
G(x) = \frac{1}{\sqrt{5}} \sum_{k \geq 0} (\phi^k - \phi^* x^k)
\]

\[
F_k = \frac{\phi^k - \phi^*}{\sqrt{5}}
\]

\[
a = \sum_{k \geq 0} ar^k x^k \quad \Rightarrow \quad \frac{1}{1 - \phi x} = \sum_{k \geq 0} \phi^k x^k \quad \frac{1}{1 - \phi^* x} = \sum_{k \geq 0} \phi^{*k} x^k
\]

\[
G(x) = \frac{1}{\sqrt{5}} \left( \frac{\phi^*}{x + \phi^*} - \frac{\phi}{x + \phi} \right) = \frac{1}{\sqrt{5}} \left( \frac{1}{1 - \phi x} - \frac{1}{1 - \phi^* x} \right)
\]
Tower of Hanoi

Object: move pile A to B by moving one disc at a time. A disc may never rest on a smaller one.

What is the minimum number of moves, $a_n$?

$a_1 = 1$, $a_2 = 3$, $a_n = a_{n-1} + 1 + a_{n-1} = 2a_{n-1} + 1 = 2^{n-1}$
\[
\frac{a}{1 - rx} = \sum_{k \geq 0} ar^k x^k
\]

\[a_n = ba_{n-1} + c \quad n > 0\]

\[G(x) = \frac{cx}{(1 - bx)(1 - x)} + \frac{a_0}{(1 - bx)}\]

\[
\frac{cx}{(1 - bx)(1 - x)} = \frac{c}{b - 1}\left(\frac{1}{1 - bx} - \frac{1}{1 - x}\right)
\]

\[G(x) = \left(a_0 + \frac{c}{b - 1}\right) \frac{1}{(1 - bx)} - \frac{c}{b - 1} \frac{1}{1 - x}\]

\[
= \left(a_0 + \frac{c}{b - 1}\right) \sum_{n \geq 0} b^n x^n - \frac{c}{b - 1} \sum_{n \geq 0} x^n
\]

\[a_n = \left(a_0 + \frac{c}{b - 1}\right) b^n - \frac{c}{b - 1}\]

**Linear recurrence relations**

**Tower of Hanoi**

\[a_0 = 0 \quad b = 2 \quad c = 1\]

\[a_n = 2^{n-1}\]
Derangements: \( D_n = (n-1)(D_{n-1} + D_{n-2}) \)

Any derangement \( f \) moves the point \( n \) to some point \( i < n \). For a fixed \( i \),

- there are \( D_{n-2} \) derangements for which \( f(i) = n \)
- if \( f(j) = n \) for \( j \neq i \), then associate to \( f \) a derangement \( f' \) on the set \( 1, 2, \ldots, n-1 \) as follows:

  \[
  f'(k) = f(k) \quad \text{for} \quad k \neq j
  \]
  \[
  f'(j) = i
  \]
More recursions

Derangements: \( D_n = (n-1)(D_{n-1} + D_{n-2}) \)

Any derangement \( f \) moves the point \( n \) to some point \( i < n \). For a fixed \( i \),

- there are \( D_{n-2} \) derangements for which \( f(i) = n \)
- if \( f(j) = n \) for \( j \neq i \), then associate to \( f \) a derangement \( f' \) on the set \( 1, 2, ..., n-1 \) as follows:
  
  \[ f'(k) = f(k) \text{ for } k \neq j \]
  
  \[ f'(j) = i \]
More recursions

Derangements: \( D_n = (n-1)(D_{n-1} + D_{n-2}) \)

Any derangement \( f \) moves the point \( n \) to some point \( i < n \). For a fixed \( i \),

- there are \( D_{n-2} \) derangements for which \( f(i) = n \)
- if \( f(j) = n \) for \( j \neq i \), then associate to \( f \) a derangement \( f' \) on the set \( 1, 2, ..., n-1 \) as follows:

  \[
  f'(k) = f(k) \text{ for } k \neq j \\
  f'(j) = i
  \]

There is a one-to-one correspondence between derangements of \( n-1 \) elements and derangements of \( n \) elements for which \( f^{-1}(n) \neq i \).
More recursions

Involutions: $v_n = v_{n-1} + (n-1)v_{n-2}$

- There are $v_{n-1}$ involutions for which $f(n) = n$
- If $f(n) = i \neq n$, then $f(i) = n$ and therefore there are $v_{n-2}$ involutions containing the cycle $(i,n)$. 
Stirling numbers of the second kind

Definition. The Stirling number of the second kind is the number of ways to partition a set of $n$ elements into $k$ non-empty subsets.

Example: $n=4$, $k=2$

$\{1,2,3\} \cup \{4\}$, $\{1,2,4\} \cup \{3\}$, $\{1,3,4\} \cup \{2\}$, $\{2,3,4\} \cup \{1\}$,

$\{1,2\} \cup \{3,4\}$, $\{1,3\} \cup \{2,4\}$, $\{1,4\} \cup \{2,3\}$

$\binom{n}{k} = 0 \quad \text{if} \quad k < 0 \quad \text{or} \quad k > n$

$\binom{n}{0} = \begin{cases} 1, & n = 0 \\ 0, & n > 0 \end{cases}$

$\binom{n}{1} = \binom{n}{n-1} = 1 \quad n \geq 1$

$\binom{n}{2} = \frac{2^n - 2}{2} = 2^{n-1} - 1 \quad n \geq 2$
Stirling numbers of the second kind

Basic recurrence relation

\[
\binom{n}{k} = k \binom{n-1}{k} + \binom{n-1}{k-1}
\]

**Proof.** We divide all partitions into two groups

A: partitions containing \( \{n\} \) as a subset,
B: all remaining partitions.

Any partition in A can be obtained from a partition of \( n-1 \) elements into \( k-1 \) subsets (there are \( S(n-1,k-1) \) such partitions) by adding to it \( \{n\} \).

Any partition in B can be obtained from a partition of \( n-1 \) elements into \( k \) subsets (\( S(n-1,k) \) partitions) by inserting the element \( n \) into one of the subsets (\( k \) subsets).
The number of surjections

**Theorem.** The number of ordered partitions of a set of $n$ elements into $k$ non-empty subsets is

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$$

**Proof.**

Let $A$ be a set of $n$ elements and $B=\{1,2,...,k\}$. Each surjection $f$ from $A$ to $B$ defines an ordered partition of $A$ into $k$ non-empty subsets $A_1,...,A_k$ as follows: $A_i=\{a \in A \mid f(a)=i\}$.

Conversely, each ordered partition of $A$ into $k$ non-empty subsets defines a surjection $f: A \rightarrow B$.

Therefore, the number of ordered partitions of $A$ coincides with the number of surjections from $A$ to $B$. 
Stirling numbers of the second kind

**Theorem.** The number of partitions of a set of $n$ elements into $k$ non-empty subsets is

$$
\begin{bmatrix} n \\ k \end{bmatrix} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n
$$
Stirling numbers of the first kind

**Definition.** The Stirling number of the first kind \( s(n, k) \) is the number of permutations of \( n \) elements consisting of \( k \) cycles.

Example: \( n=4, k=2 \)

\[
(1,2,3)(4), (1,3,2)(4), (1,2,4)(3), (1,4,2)(3), (1,3,4)(2), (1,4,3)(2),
(2,3,4)(1), (2,4,3)(1), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)
\]

\[
\begin{align*}
\left[ \begin{array}{c}
\frac{n}{k} \\
\end{array} \right] &= 0 \quad \text{if} \quad k < 0 \quad \text{or} \quad k > n \\
\left[ \begin{array}{c}
\frac{n}{0} \\
\end{array} \right] &= \begin{cases} 
1, & n = 0 \\
0, & n > 0 
\end{cases} \\
\left[ \begin{array}{c}
\frac{n}{n} \\
\end{array} \right] &= 1
\end{align*}
\]

\[
\left[ \begin{array}{c}
\frac{n}{1} \\
\end{array} \right] = (n - 1)!
\]

\[
\left[ \begin{array}{c}
\frac{n}{n - 1} \\
\end{array} \right] = \binom{n}{2}
\]

\[
\sum_{k \geq 0} \left[ \begin{array}{c}
\frac{n}{k} \\
\end{array} \right] = n!
\]
Stirling numbers of the first kind

Basic recurrence relation

\[ \left[ \begin{array}{c} n \\ k \end{array} \right] = (n-1) \left[ \begin{array}{c} n-1 \\ k \end{array} \right] + \left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right] \]

**Proof.** We divide all permutations with \( k \) cycles into two groups

A: permutations containing \( (n) \) as a cycle,
B: all remaining permutations.

Any permutation in A can be obtained from a permutation of \( n-1 \) elements with \( k-1 \) cycles (there are \( s(n-1,k-1) \) such permutations) by adding to it \( n \).

Any permutation in B can be obtained from a permutation of \( n-1 \) elements with \( k \) cycles (\( s(n-1,k) \) permutations) by inserting the element \( n \) into one of the cycles (\( n-1 \) ways).