How many labeled trees on n vertices are there?

$\frac{n^n}{n-2}$ Cayley Formula

Bijection between the set of labeled trees with n vertices and the set of all sequences of length $n-2$ composed of the elements of $\{1, 2, \ldots, n\}$.

Prüfer’s Method for Assigning a Sequence to a Labeled Tree T

1. Let $i = 0$, and let $T_0 = T$.
2. Find a vertex of degree 1 in $T_i$ with the smallest label and call it $v$.
3. Record in the sequence the label of the $v$’s neighbor.
4. Remove $v$ from $T_i$ to create a new tree $T_{i+1}$.
5. If $T_{i+1}$ has two vertices, then stop. Otherwise, increment $i$ by 1 and go back to step 2.

\[
i = 0 \quad 8 \quad 2 \quad 3 \\
4 \quad 1 \quad 5 \\
6 \quad 7 \\
\]

\[
\begin{array}{c}
2 \\
\end{array}
\]
How many labeled trees on n vertices are there?

\[ n^{n-2} \] Cayley Formula

Bijection between the set of labeled trees with n vertices and the set of all sequences of length \( n-2 \) composed of the elements of \( \{1,2,...,n\} \).

Prüfer’s Method for Assigning a Sequence to a Labeled Tree \( T \)

1. Let \( i = 0 \), and let \( T_0 = T \).
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How many labeled trees on $n$ vertices are there?

$n^{n-2}$ Cayley Formula

Bijection between the set of labeled trees with $n$ vertices and the set of all sequences of length $n-2$ composed of the elements of $\{1,2,\ldots,n\}$.

Prüfer’s Method for Assigning a Sequence to a Labeled Tree $T$

1. Let $i = 0$, and let $T_0 = T$.
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How many labeled trees on $n$ vertices are there?

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How many labeled trees on \( n \) vertices are there?

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How many labeled trees on $n$ vertices are there?

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How many labeled trees on \( n \) vertices are there?

\[ n^{n-2} \text{ Cayley Formula} \]

**Bijection between the set of labeled trees with \( n \) vertices and the set of all sequences of length \( n-2 \) composed of the elements of \( \{1, 2, \ldots, n\} \).**

**Prüfer’s Method for Assigning a Sequence to a Labeled Tree \( T \)**

1. Let \( i = 0 \), and let \( T_0 = T \).
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How many labeled trees on \( n \) vertices are there?

\[ n^{n-2} \text{ Cayley Formula} \]

Bijection between the set of labeled trees with \( n \) vertices and the set of all sequences of length \( n-2 \) composed of the elements of \( \{1,2,...,n\} \).

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3. Record in the sequence the label of the $v$’s neighbor.
4. Remove $v$ from $T_i$ to create a new tree $T_{i+1}$.
5. If $T_{i+1}$ has two vertices, then stop. Otherwise, increment $i$ by 1 and go back to step 2.

\[ i=3 \]

\[
\begin{array}{ccc}
2 & 2 & 4 \\
\end{array}
\]
How many labeled trees on $n$ vertices are there?

$n^{n-2}$ Cayley Formula

Bijection between the set of labeled trees with $n$ vertices and the set of all sequences of length $n-2$ composed of the elements of $\{1,2,\ldots,n\}$.

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1. Let $i = 0$, and let $T_0 = T$.
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How many labeled trees on \( n \) vertices are there?

\[ n^{n-2} \] Cayley Formula

Bijection between the set of labeled trees with \( n \) vertices and the set of all sequences of length \( n-2 \) composed of the elements of \( \{1,2,...,n\} \).

Prüfer’s Method for Assigning a Sequence to a Labeled Tree \( T \)

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5. If \( T_{i+1} \) has two vertices, then stop. Otherwise, increment \( i \) by 1 and go back to step 2.

\[ i=3 \]

\[
\begin{array}{cccc}
2 & 2 & 4 & 4 \\
\end{array}
\]
How many labeled trees on \( n \) vertices are there?

\[ n^{n-2} \quad \text{Cayley Formula} \]

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3. Record in the sequence the label of the \( v \)'s neighbor.
4. Remove \( v \) from \( T_i \) to create a new tree \( T_{i+1} \).
5. If \( T_{i+1} \) has two vertices, then stop. Otherwise, increment \( i \) by 1 and go back to step 2.

\[ i=4 \]

\[ \begin{array}{cccc} 2 & 2 & 4 & 4 \end{array} \]

\( i = 4 \)

\( \begin{array}{c} 8 \end{array} \)

\( \begin{array}{c} 4 \end{array} \)

\( \begin{array}{c} 1 \end{array} \)

\( \begin{array}{c} 7 \end{array} \)
How many labeled trees on \( n \) vertices are there?

\[ n^{n-2} \] Cayley Formula

Bijection between the set of labeled trees with \( n \) vertices and the set of all sequences of length \( n-2 \) composed of the elements of \( \{1, 2, \ldots, n\} \).

Prüfer’s Method for Assigning a Sequence to a Labeled Tree \( T \)

1. Let \( i = 0 \), and let \( T_0 = T \).
2. Find a vertex of degree 1 in \( T_i \) with the smallest label and call it \( v \).
3. Record in the sequence the label of the \( v \)’s neighbor.
4. Remove \( v \) from \( T_i \) to create a new tree \( T_{i+1} \).
5. If \( T_{i+1} \) has two vertices, then stop. Otherwise, increment \( i \) by 1 and go back to step 2.

\[ i=4 \]

\[
\begin{array}{c}
2 \\
2 \\
4 \\
4 \\
1
\end{array}
\]
How many labeled trees on \( n \) vertices are there?

\[ n^{n-2} \quad \text{Cayley Formula} \]

Bijection between the set of labeled trees with \( n \) vertices and the set of all sequences of length \( n-2 \) composed of the elements of \( \{1, 2, \ldots, n\} \).

Prüfer’s Method for Assigning a Sequence to a Labeled Tree \( T \)

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2. Find a vertex of degree 1 in \( T_i \) with the smallest label and call it \( v \).
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4. Remove \( v \) from \( T_i \) to create a new tree \( T_{i+1} \).
5. If \( T_{i+1} \) has two vertices, then stop. Otherwise, increment \( i \) by 1 and go back to step 2.

\( i = 4 \)

\[
\begin{array}{c}
8 \\
4 \\
1 \\
\end{array}
\]

\[
\begin{array}{c}
2 \\
2 \\
4 \\
4 \\
1 \\
\end{array}
\]
How many labeled trees on n vertices are there?

\[ n^{n-2} \]

Cayley Formula

Bijection between the set of labeled trees with n vertices and the set of all sequences of length \( n-2 \) composed of the elements of \( \{1,2,\ldots,n\} \).

Prüfer’s Method for Assigning a Sequence to a Labeled Tree T

1. Let \( i = 0 \), and let \( T_0 = T \).
2. Find a vertex of degree 1 in \( T_i \) with the smallest label and call it \( v \).
3. Record in the sequence the label of the \( v \)’s neighbor.
4. Remove \( v \) from \( T_i \) to create a new tree \( T_{i+1} \).
5. If \( T_{i+1} \) has two vertices, then stop. Otherwise, increment \( i \) by 1 and go back to step 2.

i=5

\[ \begin{array}{c}
8 \\
4 \\
1 \\
\end{array} \]

\[ \begin{array}{ccccc}
2 & 2 & 4 & 4 & 1 \\
\end{array} \]
How many labeled trees on $n$ vertices are there?

$\frac{n^n}{n-2}$ Cayley Formula

Bijection between the set of labeled trees with $n$ vertices and the set of all sequences of length $n-2$ composed of the elements of $\{1,2,\ldots,n\}$.

Prüfer’s Method for Assigning a Sequence to a Labeled Tree $T$

1. Let $i = 0$, and let $T_0 = T$.
2. Find a vertex of degree 1 in $T_i$ with the smallest label and call it $v$.
3. Record in the sequence the label of the $v$’s neighbor.
4. Remove $v$ from $T_i$ to create a new tree $T_{i+1}$.
5. If $T_{i+1}$ has two vertices, then stop. Otherwise, increment $i$ by 1 and go back to step 2.

$\begin{array}{cccccc}
8 & 4 & 4 & 4 & 1 & 4 \\
\end{array}$
How many labeled trees on n vertices are there?

$n^{n-2}$ Cayley Formula

Bijection between the set of labeled trees with n vertices and the set of all sequences of length \(n-2\) composed of the elements of \{1,2,...,n\}.

Prüfer’s Method for Assigning a Sequence to a Labeled Tree T

1. Let $i = 0$, and let $T_0 = T$.
2. Find a vertex of degree 1 in $T_i$ with the smallest label and call it $v$.
3. Record in the sequence the label of the $v$'s neighbor.
4. Remove $v$ from $T_i$ to create a new tree $T_{i+1}$.
5. If $T_{i+1}$ has two vertices, then stop. Otherwise, increment $i$ by 1 and go back to step 2.

$i=5$

\[
\begin{array}{c}
2 & 2 & 4 & 4 & 1 & 4
\end{array}
\]
How many labeled trees on n vertices are there?

\[ n^{n-2} \] Cayley Formula

Bijection between the set of labeled trees with n vertices and the set of all sequences of length \( n - 2 \) composed of the elements of \{1,2,...,n\}.

Prüfer’s Method for Assigning a Sequence to a Labeled Tree \( T \)

1. Let \( i = 0 \), and let \( T_0 = T \).
2. Find a vertex of degree 1 in \( T_i \) with the smallest label and call it \( v \).
3. Record in the sequence the label of the \( v \)’s neighbor.
4. Remove \( v \) from \( T_i \) to create a new tree \( T_{i+1} \).
5. If \( T_{i+1} \) has two vertices, then stop. Otherwise, increment \( i \) by 1 and go back to step 2.

\[ \begin{array}{c}
1 \ 8 \\
4 \\
\end{array} \]

The number of trees \( \leq \) the number of sequences
How many labeled trees on $n$ vertices are there?

$n^{n-2}$ Cayley Formula

Bijection between the set of labeled trees with $n$ vertices and the set of all sequences of length $n-2$ composed of the elements of $\{1,2,...,n\}$.

Prüfer’s Method for Assigning a Labeled Tree with vertex set $\{1,...,n\}$ to a Sequence $\sigma=(j_1,...,j_{n-2})$, where $j_i \in \{1,...,n\}$ for each $i=1,...,n-2$.

1. Let $i = 0$, let $\sigma_0 = \sigma$ and let $S_0 = \{1,2,...,n\}$.
2. Let $j$ be the smallest number in $S_i$ that does not appear in the sequence $\sigma_i$.
3. Connect vertex $j$ to vertex $j_{i+1}$ (the first element of $\sigma_i$).
4. Remove the first element of $\sigma_i$ to create a new sequence $\sigma_{i+1}$, and remove $j$ from $S_i$ to create a new set $S_{i+1}$.
5. If the sequence $\sigma_{i+1}$ is empty, connect the last two vertices in $S_{i+1}$ and stop. Otherwise, increment $i$ by 1 and go back to step 2.

$S_0 = \{1,2,3,4,5,6,7,8\}$ $i=0$

$\begin{bmatrix} 2 & 2 & 4 & 4 & 1 & 4 \end{bmatrix} = \sigma_0$

The number of trees $\leq$ the number of sequences
How many labeled trees on n vertices are there?

\[ n^{n-2} \] Cayley Formula

Bijection between the set of labeled trees with n vertices and the set of all sequences of length \( n-2 \) composed of the elements of \( \{1,2,\ldots,n\} \).

Prüfer’s Method for Assigning a Labeled Tree with vertex set \( \{1,\ldots,n\} \) to a Sequence \( \sigma=(j_1,\ldots,j_{n-2}) \), where \( j_i{\in}\{1,\ldots,n\} \) for each \( i=1,\ldots,n-2 \).

1. Let \( i=0 \), let \( \sigma_0=\sigma \) and let \( S_0=\{1,2,\ldots,n\} \).
2. Let \( j \) be the smallest number in \( S_i \) that does not appear in the sequence \( \sigma_i \).
3. Connect vertex \( j \) to vertex \( j_{i+1} \) (the first element of \( \sigma_i \)).
4. Remove the first element of \( \sigma_i \) to create a new sequence \( \sigma_{i+1} \), and remove \( j \) from \( S_i \) to create a new set \( S_{i+1} \).
5. If the sequence \( \sigma_{i+1} \) is empty, connect the last two vertices in \( S_{i+1} \) and stop. Otherwise, increment \( i \) by 1 and go back to step 2.

\[ S_0=\{1,2,3,4,5,6,7,8\} \quad i=0 \]

\[ \begin{array}{ccccccc} 2 & 2 & 4 & 4 & 1 & 4 \end{array} = \sigma_0 \]

The number of trees \( \leq \) the number of sequences.
How many labeled trees on \( n \) vertices are there?

\[ n^{n-2} \] Cayley Formula

Bijection between the set of labeled trees with \( n \) vertices and the set of all sequences of length \( n-2 \) composed of the elements of \( \{1,2,\ldots,n\} \).

Prüfer’s Method for Assigning a Labeled Tree with vertex set \( \{1,\ldots,n\} \) to a Sequence \( \sigma=(j_1,\ldots,j_{n-2}) \), where \( j_i\in\{1,\ldots,n\} \) for each \( i=1,\ldots,n-2 \).

1. Let \( i = 0 \), let \( \sigma_0=\sigma \) and let \( S_0=\{1,2,\ldots,n\} \).
2. Let \( j \) be the smallest number in \( S_i \) that does not appear in the sequence \( \sigma_i \).
3. Connect vertex \( j \) to vertex \( j_{i+1} \) (the first element of \( \sigma_i \)).
4. Remove the first element of \( \sigma_i \) to create a new sequence \( \sigma_{i+1} \), and remove \( j \) from \( S_i \) to create a new set \( S_{i+1} \).
5. If the sequence \( \sigma_{i+1} \) is empty, connect the last two vertices in \( S_{i+1} \) and stop. Otherwise, increment \( i \) by 1 and go back to step 2.

The number of trees \( \leq \) the number of sequences.
How many labeled trees on n vertices are there?

\[ n^{n-2} \]

Cayley Formula

Bijection between the set of labeled trees with n vertices and the set of all sequences of length n\(−2\) composed of the elements of \(\{1,2,\ldots,n\}\).

Prüfer’s Method for Assigning a Labeled Tree with vertex set \(\{1,\ldots,n\}\) to a Sequence \(\sigma=(j_1,\ldots,j_{n-2})\), where \(j_i\in\{1,\ldots,n\}\) for each \(i=1,\ldots,n-2\).

1. Let \(i=0\), let \(\sigma_0=\sigma\) and let \(S_0=\{1,2,\ldots,n\}\).
2. Let \(j\) be the smallest number in \(S_i\) that does not appear in the sequence \(\sigma_i\).
3. Connect vertex \(j\) to vertex \(j_{i+1}\) (the first element of \(\sigma_i\)).
4. Remove the first element of \(\sigma_i\) to create a new sequence \(\sigma_{i+1}\), and remove \(j\) from \(S_i\) to create a new set \(S_{i+1}\).
5. If the sequence \(\sigma_{i+1}\) is empty, connect the last two vertices in \(S_{i+1}\) and stop. Otherwise, increment \(i\) by 1 and go back to step 2.

\[
\begin{array}{c}
2 \\
\hline
3
\end{array}
\]

\[
S_1=\{1,2,4,5,6,7,8\} \quad i=1
\]

\[
\begin{array}{cccccc}
2 & 4 & 4 & 1 & 4
\end{array} = \sigma_1
\]

The number of trees \(\leq\) the number of sequences
How many labeled trees on $n$ vertices are there?

$n^{n-2}$ Cayley Formula

Prüfer’s Method for Assigning a Labeled Tree with vertex set \{1,...,n\} to a Sequence $\sigma=(j_1,...,j_{n-2})$, where $j_i \in \{1,...,n\}$ for each $i=1,...,n-2$.

1. Let $i = 0$, let $\sigma_0=\sigma$ and let $S_0=\{1,2,...,n\}$.
2. Let $j$ be the smallest number in $S_i$ that does not appear in the sequence $\sigma_i$.
3. Connect vertex $j$ to vertex $j_{i+1}$ (the first element of $\sigma_i$).
4. Remove the first element of $\sigma_i$ to create a new sequence $\sigma_{i+1}$, and remove $j$ from $S_i$ to create a new set $S_{i+1}$.
5. If the sequence $\sigma_{i+1}$ is empty, connect the last two vertices in $S_{i+1}$ and stop. Otherwise, increment $i$ by 1 and go back to step 2.

$S_1=\{1,2,4,5,6,7,8\} \quad i=1$

$\begin{bmatrix} 2 & 4 & 4 & 1 & 4 \end{bmatrix} = \sigma_1$

The number of trees $\leq$ the number of sequences
How many labeled trees on $n$ vertices are there?

$n^{n-2}$ Cayley Formula

Bijection between the set of labeled trees with $n$ vertices and the set of all sequences of length $n-2$ composed of the elements of $\{1,2,...,n\}$.

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2. Let $j$ be the smallest number in $S_i$ that does not appear in the sequence $\sigma_i$.
3. Connect vertex $j$ to vertex $j_{i+1}$ (the first element of $\sigma_i$).
4. Remove the first element of $\sigma_i$ to create a new sequence $\sigma_{i+1}$, and remove $j$ from $S_i$ to create a new set $S_{i+1}$.
5. If the sequence $\sigma_{i+1}$ is empty, connect the last two vertices in $S_{i+1}$ and stop. Otherwise, increment $i$ by 1 and go back to step 2.

The number of trees $\leq$ the number of sequences
How many labeled trees on \( n \) vertices are there?

\[ n^{n-2} \quad \text{Cayley Formula} \]

Bijection between the set of labeled trees with \( n \) vertices and the set of all sequences of length \( n-2 \) composed of the elements of \{1,2,...,n\}.

Prüfer’s Method for Assigning a Labeled Tree with vertex set \{1,...,n\} to a Sequence \( \sigma=(j_1,...,j_{n-2}) \), where \( j_i \in \{1,...,n\} \) for each \( i=1,...,n-2 \).

1. Let \( i = 0 \), let \( \sigma_0=\sigma \) and let \( S_0=\{1,2,...,n\} \).
2. Let \( j \) be the smallest number in \( S_i \) that does not appear in the sequence \( \sigma_i \).
3. Connect vertex \( j \) to vertex \( j_{i+1} \) (the first element of \( \sigma_i \)).
4. Remove the first element of \( \sigma_i \) to create a new sequence \( \sigma_{i+1} \), and remove \( j \) from \( S_i \) to create a new set \( S_{i+1} \).
5. If the sequence \( \sigma_{i+1} \) is empty, connect the last two vertices in \( S_{i+1} \) and stop. Otherwise, increment \( i \) by 1 and go back to step 2.

\[
S_2=\{1,2,4,6,7,8\} \quad \text{i}=2
\]

\[
\begin{array}{cccc}
4 & 4 & 1 & 4 \\
\end{array}
=\sigma_2
\]

The number of trees \( \leq \) the number of sequences
How many labeled trees on \( n \) vertices are there?

\[ n^{n-2} \text{ Cayley Formula} \]

**Bijection between the set of labeled trees with \( n \) vertices and the set of all sequences of length \( n-2 \) composed of the elements of \( \{1,2,...,n\} \).**

Prüfer’s Method for Assigning a Labeled Tree with vertex set \( \{1,...,n\} \) to a Sequence \( \sigma=(j_1,...,j_{n-2}) \), where \( j_i\in\{1,...,n\} \) for each \( i=1,...,n-2 \).

1. Let \( i = 0 \), let \( \sigma_0=\sigma \) and let \( S_0=\{1,2,...,n\} \).
2. Let \( j \) be the smallest number in \( S_i \) that does not appear in the sequence \( \sigma_i \).
3. Connect vertex \( j \) to vertex \( j_{i+1} \) (the first element of \( \sigma_i \)).
4. Remove the first element of \( \sigma_i \) to create a new sequence \( \sigma_{i+1} \), and remove \( j \) from \( S_i \) to create a new set \( S_{i+1} \).
5. If the sequence \( \sigma_{i+1} \) is empty, connect the last two vertices in \( S_{i+1} \) and stop. Otherwise, increment \( i \) by 1 and go back to step 2.

![](tree_diagram.png)

\[ S_2=\{1,2,4,6,7,8\} \quad i=2 \]

\[ \begin{array}{c|c|c|c|c}
\hline
& 4 & 4 & 1 & 4 \\
\hline
\end{array} = \sigma_2 \]

The number of trees \( \leq \) the number of sequences
How many labeled trees on $n$ vertices are there?

$n^{n-2}$ Cayley Formula

Bijection between the set of labeled trees with $n$ vertices and the set of all sequences of length $n-2$ composed of the elements of $\{1,2,\ldots,n\}$.

Prüfer’s Method for Assigning a Labeled Tree with vertex set $\{1,\ldots,n\}$ to a Sequence $\sigma=(j_1,\ldots,j_{n-2})$, where $j_i \in \{1,\ldots,n\}$ for each $i=1,\ldots,n-2$.

1. Let $i = 0$, let $\sigma_0 = \sigma$ and let $S_0 = \{1,2,\ldots,n\}$.
2. Let $j$ be the smallest number in $S_i$ that does not appear in the sequence $\sigma_i$.
3. Connect vertex $j$ to vertex $j_{i+1}$ (the first element of $\sigma_i$).
4. Remove the first element of $\sigma_i$ to create a new sequence $\sigma_{i+1}$, and remove $j$ from $S_i$ to create a new set $S_{i+1}$.
5. If the sequence $\sigma_{i+1}$ is empty, connect the last two vertices in $S_{i+1}$ and stop. Otherwise, increment $i$ by 1 and go back to step 2.

$S_2 = \{1, 4, 6, 7, 8\}$ \quad i=2

The number of trees $\leq$ the number of sequences
How many labeled trees on \( n \) vertices are there?

\[ n^{n-2} \quad \text{Cayley Formula} \]

Bijection between the set of labeled trees with \( n \) vertices and the set of all sequences of length \( n-2 \) composed of the elements of \{1,2,...,n\}.

Prüfer’s Method for Assigning a Labeled Tree with vertex set \{1,...,n\} to a Sequence \( \sigma=(j_1,...,j_{n-2}) \), where \( j_i \in \{1,...,n\} \) for each \( i=1,...,n-2 \).

1. Let \( i = 0 \), let \( \sigma_0=\sigma \) and let \( S_0=\{1,2,...,n\} \).
2. Let \( j \) be the smallest number in \( S_i \) that does not appear in the sequence \( \sigma_i \).
3. Connect vertex \( j \) to vertex \( j_{i+1} \) (the first element of \( \sigma_i \)).
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Bijection between the set of labeled trees with \( n \) vertices and the set of all sequences of length \( n-2 \) composed of the elements of \( \{1,2,\ldots,n\} \).

Prüfer’s Method for Assigning a Labeled Tree with vertex set \( \{1,\ldots,n\} \) to a Sequence \( \sigma=(j_1,\ldots,j_{n-2}) \), where \( j_i \in \{1,\ldots,n\} \) for each \( i=1,\ldots,n-2 \).
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\[ i=3 \quad S_3=\{1, \ , \ , 4, \ , 6,7,8\} \]

The number of trees \( \leq \) the number of sequences
How many labeled trees on n vertices are there?

\[ n^{n-2} \] Cayley Formula

Bijection between the set of labeled trees with n vertices and the set of all sequences of length \( n-2 \) composed of the elements of \{1,2,...,n\}.

Prüfer’s Method for Assigning a Labeled Tree with vertex set \{1,...,n\} to a Sequence \( \sigma=(j_1,...,j_{n-2}) \), where \( j_i \in \{1,...,n\} \) for each \( i=1,...,n-2 \).

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5. If the sequence \( \sigma_{i+1} \) is empty, connect the last two vertices in \( S_{i+1} \) and stop. Otherwise, increment \( i \) by 1 and go back to step 2.

\[ S_3=\{1, ,4, ,7,8\} \quad i=3 \]

The number of trees \( \leq \) the number of sequences
How many labeled trees on n vertices are there?

\[ n^{n-2} \quad \text{Cayley Formula} \]

Prüfer’s Method for Assigning a Labeled Tree with vertex set \{1,\ldots,n\} to a Sequence \( \sigma = (j_1, \ldots, j_{n-2}) \), where \( j_i \in \{1, \ldots, n\} \) for each \( i = 1, \ldots, n-2 \).
1. Let \( i = 0 \), let \( \sigma_0 = \sigma \) and let \( S_0 = \{1,2,\ldots,n\} \).
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\[
S_4=\{1,\ ,\ ,4,\ ,\ ,7,8\} \quad i=4
\]

\[
\begin{array}{c|c}
& 1 & 4 \\
\hline
1 & & \\
\end{array}
= \sigma_4
\]

The number of trees \( \leq \) the number of sequences
How many labeled trees on $n$ vertices are there?

$n^{n-2}$ Cayley Formula

Bijection between the set of labeled trees with $n$ vertices and the set of all sequences of length $n-2$ composed of the elements of $\{1,2,\ldots,n\}$.

Prüfer’s Method for Assigning a Labeled Tree with vertex set $\{1,\ldots,n\}$ to a Sequence $\sigma=(j_1,\ldots,j_{n-2})$, where $j_i \in \{1,\ldots,n\}$ for each $i=1,\ldots,n-2$.

1. Let $i=0$, let $\sigma_0=\sigma$ and let $S_0=\{1,2,\ldots,n\}$.
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$S_4=\{1,\ ,\ ,4,\ ,\ ,7,8\}$ \quad i=4

The number of trees $\leq$ the number of sequences
How many labeled trees on \( n \) vertices are there?

\[ n^{n-2} \text{ Cayley Formula} \]

Bijection between the set of labeled trees with \( n \) vertices and the set of all sequences of length \( n-2 \) composed of the elements of \( \{1,2,\ldots,n\} \).

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The number of trees \( \leq \) the number of sequences
How many labeled trees on $n$ vertices are there?

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Bijection between the set of labeled trees with $n$ vertices and the set of all sequences of length $n-2$ composed of the elements of $\{1,2,\ldots,n\}$.

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$S_5 = \{1, , 4, , , , 8\}$ \hspace{1cm} i=5

The number of trees $\leq$ the number of sequences
How many labeled trees on n vertices are there?

\[ n^{n-2} \text{ Cayley Formula} \]

**Bijection between the set of labeled trees with n vertices and the set of all sequences of length n−2 composed of the elements of \{1,2,...,n\}.**

Prüfer’s Method for Assigning a Labeled Tree with vertex set \{1,...,n\} to a Sequence \( \sigma=(j_1,...,j_{n-2}) \), where \( j_i \in \{1,...,n\} \) for each \( i=1,...,n-2 \).

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\[ S_5 = \{1, , ,4, , , ,8\} \quad i=5 \]

The number of trees \( \leq \) the number of sequences.
How many labeled trees on \( n \) vertices are there?

\[ n^{n-2} \quad \text{Cayley Formula} \]

Bijection between the set of labeled trees with \( n \) vertices and the set of all sequences of length \( n-2 \) composed of the elements of \( \{1,2,...,n\} \).

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\[ S_5=\{ , , ,4, , , ,8\} \quad i=5 \]

The number of trees \( \leq \) the number of sequences
How many labeled trees on \( n \) vertices are there?

\[ n^{n-2} \quad \text{Cayley Formula} \]

There is a bijection between the set of labeled trees with \( n \) vertices and the set of all sequences of length \( n-2 \) composed of the elements of \{1,2,...,n\}.

Prüfer’s Method for Assigning a Labeled Tree with vertex set \{1,...,n\} to a Sequence \( \sigma=(j_1,...,j_{n-2}) \), where \( j_i \in \{1,...,n\} \) for each \( i=1,...,n-2 \).

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\[
S_5 = \{ , , ,4, , , ,8\} \quad i=5
\]

The number of trees \( \leq \) the number of sequences
How many labeled trees on \( n \) vertices are there?

\[ n^{n-2} \] Cayley Formula

Prüfer’s Method for Assigning a Labeled Tree with vertex set \( \{1,...,n\} \) to a Sequence \( \sigma=(j_1,...,j_{n-2}) \), where \( j_i \in \{1,...,n\} \) for each \( i=1,...,n-2 \).

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5. If the sequence \( \sigma_{i+1} \) is empty, connect the last two vertices in \( S_{i+1} \) and stop. Otherwise, increment \( i \) by 1 and go back to step 2.

\[ S_5=\{ , , ,4, , , ,8\} \quad i=5 \]

The number of trees = the number of sequences

Bijection between the set of labeled trees with \( n \) vertices and the set of all sequences of length \( n-2 \) composed of the elements of \( \{1,2,...,n\} \).
How many labeled trees on n vertices are there?

Exercise. Find the tree associated to the sequence
(5, 1, 7, 2)
(1, 1, 1, ..., 1)
How many unlabeled trees on n vertices are there?

**Theorem.** The number of pairwise non-isomorphic trees with n vertices is at most $4^{n-1}$

**Definition.** A rooted tree is a tree with a designated vertex, the root of the tree.

00101100010111

This procedure maps a rooted tree with m edges into a binary word of length 2m. Different (non-isomorphic) rooted trees are mapped into different words. The number of non-isomorphic rooted trees $\leq$ the number of binary words of length 2m $= 2^m = 4^m = 4^{n-1}$
**Definition.** The distance from the root to the farthest leaf is the height the tree.

**Definition.** A binary tree is a rooted tree in which every vertex has at most 2 children.

**Definition.** A strictly binary tree is a binary tree in which every non-leaf vertex has exactly 2 children (full binary tree).

**Definition.** A complete binary tree is a strictly binary tree in which all leaves are of the same distance from the root.
Rooted trees

Exercises.

What is the maximum number of vertices in a binary tree of height $h$?

What is the minimum height of a binary tree with $n$ leaves?