# Stochastic Calculus and Applications (L24)

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Stochastic Calculus is an extension of classical calculus for functions of a single variable, which applies in particular to almost all functions arising as a path of Brownian motion, even though such paths are nowhere differentiable. The key result is Itô's formula which is a sort of chain rule. There are two important classes of continuous-time processes in $\mathbb{R}^d$: those which evolve by a discrete series of jumps and those which look locally like Brownian motion. There is a stochastic calculus associated to both classes of process which provides a powerful analytical tool in their study. The course will develop this approach and give examples of its application.

Those attending this course will normally also attend the Part III course Advanced Probability, which covers all the prerequisite material. A prior acquaintance with Brownian motion, continuous-time Markov chains and martingale theory is highly desirable, as given, for example, in Kallenberg's book, Chapters 6, 10, 11.

- *Stochastic calculus for continuous martingales*
  Martingales and local martingales. The Hilbert space $\mathcal{M}^2$ of $L^2$-bounded martingales. Finite variation processes: total variation, Lebesgue–Stieltjes integral. Any continuous local martingale of finite variation is constant. Adaptedness and previsibility. Stochastic integrals I: $H \in \mathcal{S}$, $M \in \mathcal{M}_2$. Quadratic variation in $\mathcal{M}^c_{\text{loc}}$. Stochastic integrals II: $H \in L^2(M)$, $M \in \mathcal{M}^c_2$, extension by localization, basic properties, approximation by Riemann sums. Covariation in $\mathcal{M}^c_{\text{loc}}$, Kunita–Watanabe identity. Semimartingales, Doob–Meyer decomposition. Itô formula. Stratonovich integrals. Differential calculus. Exponentials.

- *Stochastic calculus for jump processes*
  Stochastic integration with respect to an integer-valued random measure. Poisson random measures, construction of Lévy processes. Pure jump Markov processes in $\mathbf{R}^d$, Lévy kernel, Kurtz' theorem for the fluid limit.

- *Stochastic differential equations*
  Stochastic differential equations driven by Brownian motion. Existence and uniqueness for Lipschitz coefficients. Examples: Brownian exponential, Ornstein–Uhlenbeck process, noisy dynamical system, Bessel processes. Local existence and uniqueness for locally Lipschitz coefficients. Relation with second order elliptic and parabolic partial differential equations: Dirichlet problem and Cauchy problem. Feynman–Kač formula. Diffusion processes: $L$-diffusions, strong Markov property, construction via stochastic differential equations, identification of finite-dimensional distributions in terms of the heat kernel.

- *Applications*
  Lévy's characterization of Brownian motion; identification of Bessel processes with the radial part of Brownian motion, identification of the Ornstein–Uhlenbeck transition density. Continuous local martingales as time-changes of Brownian motion. Exponential martingale inequality. Girsanov's theorem, Cameron–Martin formula.

**Level**: Additional

### Books


B. Oksendal, *Stochastic Differential Equations: an introduction with applications*, Springer, 1992.

O. Kallenberg, *Foundations of Modern Probability*, Springer (1997). Chapters 15, 16, 18, 21.

D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer, 1991.

L. C. G. Rogers and D. Williams, *Diffusions, Markov Processes and Martingales, Vol 2: Itô calculus*, Wiley, 1987.

D. W. Stroock, *Probability Theory: an analytic view*, C.U.P., 1994.