Stochastic Calculus and Applications — Lent 2007

Example Sheet 3

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Girsanov's theorem

1. Let B be a standard Brownian motion and for $a, b > 0$ let $\tau_{a,b} = \inf\{t \geq 0 : B_t + bt = a\}$. Use Girsanov's theorem to prove that the density of $\tau_{a,b}$ is

$$a(2\pi t^3)^{-1/2} \exp(-(a - bt)^2/2t).$$

2. Suppose M is a continuous local martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ such that $[M]$ is uniformly bounded. Let $Z_t = \exp(M_t - \frac{1}{2}[M]_t)$ and let

$$\tilde{\mathbb{P}}(A) = \mathbb{E}\left[Z_\infty \mathbb{1}_A\right].$$

Show, using Girsanov's theorem, that any $\mathbb{P}$-semimartingale must also be a $\tilde{\mathbb{P}}$-semimartingale and vice versa.

Stochastic differential equations

3. Consider the stochastic differential equation with Lipschitz coefficients

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt, \quad X_0 = x_0.$$

Show that there is uniqueness in law (i.e. that the pathwise unique solutions on different spaces given by Theorem 5.3.1 must have the same distribution).

4. [Non-examinable] Consider the stochastic differential equation

$$dX_t = \mathrm{sgn}(X_t)dB_t, \quad X_0 = 0,$$

discussed in Example 5.2.1. By Lévy's characterization, any solution must be a standard Brownian motion. Suppose that X is such a solution. Show, using Tanaka's formula (Example Sheet 2, Question 11), that the natural filtration of B coincides with the natural filtration of $|X|$. Deduce that there does not exist a strong solution to the SDE.

5. Suppose that σ, b and $\sigma_n, b_n, n = 1, 2, \ldots$ are Lipschitz, with constant K independent of n. Suppose also that $\sigma_n \to \sigma$ and $b_n \to b$ uniformly. Define X and X^n by

$$\begin{aligned} dX_t &= \sigma(X_t)dB_t + b(X_t)dt, & X_0 &= x, \\ dX_t^n &= \sigma_n(X_t^n)dB_t + b_n(X_t^n)dt, & X_0^n &= x. \end{aligned}$$

Show that, as $n \to \infty$,

$$\mathbb{E}_x\left[\sup_{s\leq t} |X_s^n - X_s|^2\right] \to 0.$$

6. Let B be a standard Brownian motion on $\mathbb{R}$. Use the integration by parts formula to give a simple proof that $\mathcal{E}(\lambda B)_t = x_0 \exp(\lambda B_t - \frac{\lambda^2}{2} t)$ is the pathwise unique solution to the SDE
$$dX_t = \lambda X_t dB_t, \quad X_0 = x_0.$$
[Hint: if Y is another solution, prove that $d(YX^{-1}) = 0$.]

7. Suppose that X satisfies the stochastic differential equation
$$dX_t = \sigma(X_t)dB_t, \quad X_0 = x_0,$$
where σ is Lipschitz and B is a Brownian motion. Show that, for some constant $C < \infty$ depending only on the Lipschitz constant of σ, X satisfies the estimate
$$\mathbb{E}\left[\sup_{s\le t} |X_s - x_0|^2\right] \le Cte^{Ct}|\sigma(x_0)|^2.$$
Discuss the tightness of this estimate with reference to the special cases $\sigma(x) = 1$ and $\sigma(x) = x$.

8. Consider the stochastic differential equation in $\mathbb{R}$
$$dX_t = dB_t + b(X_t)dt, \quad X_0 = x,$$
where b is bounded and measurable. Suppose that, under $\mathbb{P}$, X is a Brownian motion started from x. Use Girsanov's theorem to find a new probability measure $\tilde{\mathbb{P}}$, absolutely continuous with respect to $\mathbb{P}$, such that if
$$B_t = X_t - \int_0^t b(X_s)ds,$$
then $(B_t)_{0\le t\le 1}$ is a Brownian motion under $\tilde{\mathbb{P}}$. This is called solution by *transformation of drift.*

9. Consider the Ornstein-Uhlenbeck system
$$\begin{aligned} dV_t &= dB_t - \lambda V_t dt, \qquad & V_0 &= 0, \\ dX_t &= V_t dt, & X_0 &= 0. \end{aligned}$$
Find the joint distribution of (X_t, V_t).

10. Let B be standard Brownian motion on $\mathbb{R}$ and $(\mathcal{F}_t)_{t\ge 0}$ its natural filtration. For $\lambda > 0$, define $X_t = B_{\exp(2\lambda t)}$ and $\mathcal{G}_t = \mathcal{F}_{\exp(2\lambda t)}$.

    (a) Show that X is a $(\mathcal{G}_t)$-martingale and compute its quadratic variation process.

    (b) Write X_t as a stochastic integral with respect to a $(\mathcal{G}_t)$-adapted Brownian motion W (which you should construct).

    (c) Conclude that the process $Y_t = e^{-\lambda t} B_{\exp(2\lambda t)}$ satisfies the Ornstein-Uhlenbeck stochastic differential equation
    $$dY_t = \sqrt{2\lambda}\, dW_t - \lambda Y_t\, dt,$$
    and solve the equation to express Y_t explicitly in terms of W_t.

11. Let $X_t = t + (1-t)\int_0^t \frac{dB_s}{1-s}$, $0 \le t \le 1$.

(a) Prove that X solves the SDE

$$dX_t = dB_t + \frac{1 - X_t}{1-t}dt, \quad X_0 = 0, \quad t \in [0,1].$$

(b) Prove that X is a Gaussian process with $\mathbb{E}[X_t] = t$ and $\operatorname{cov}(X_s, X_t) = s \wedge t - st$.

(c) Set $Y_t = B_t + t(1 - B_1), 0 \le t \le 1$, and show that Y has the same distribution as X.

X is called the *Brownian bridge* from 0 to 1.

12. Consider the stochastic differential equation

$$dX_t = \sigma(X_t)dB_t, \quad X_0 = 0$$

where $\sigma : \mathbb{R} \to (0, \infty)$ is locally Lipschitz and $\sigma \equiv 1$ on $(-\infty, 0]$. Show by expressing the solution X as a time change of Brownian motion that X cannot explode.