Stochastic Calculus and Applications **Lent 2007**

Example Sheet 2

Please send corrections to Stefan Grosskinsky <stefan@statslab.cam.ac.uk>.

Stochastic integration

1. Let $(\Omega, \mathbb{P})$ be a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ and let $X_t(\omega)$ be a continuous $((F_t)_{t\geq 0}, \mathbb{P})$-semimartingale. Show that the quadratic variation process $[X]$ does not depend on the filtration. [Use the fact that $[X]_t$ is the u.c.p. limit of the sums $\sum_{k=0}^{\lfloor 2^n t\rfloor -1}(X_{(k+1)2^{-n}} - X_{k2^{-n}})^2$].
   Likewise, prove that if $\mathbb{Q}$ is a probability measure such that $\mathbb{Q} \ll \mathbb{P}$ and X is still a $\mathbb{Q}$-semimartingale, then the quadratic variation is the same, regardless of which measure ($\mathbb{P}$ or $\mathbb{Q}$) we use to compute it.

2. (A stochastic version of the Lebesgue dominated convergence theorem.) Let X be a continuous semimartingale. If $\{H^n, n \geq 1\}$ is a sequence of locally bounded processes converging to zero pointwise and if there exists a locally bounded process H such that $|H^n| \leq H$ for every n, then $H^n \cdot X$ converges to zero u.c.p.
   [Hints: Treat the finite variation and local martingale parts separately. For the local martingale part, let (T_k) be a sequence of stopping times with $T_k \uparrow \infty$ which reduces X and which is such that $H\mathbb{1}_{(0,T_k]}$ is bounded for all k. Then $H^n\mathbb{1}_{(0,T_k]}$ converges to zero in $L^2(X^{T_k})$ and, therefore, $(H^n \cdot X)^{T_k}$ converges to zero in $\mathcal{M}_c^2$.]

Stochastic calculus and Itô's formula

3. Let B be a Brownian motion in $\mathbb{R}$ and let X be a continuous semimartingale. Suppose that $\sigma, b : \mathbb{R} \to \mathbb{R}$ are C^3. Show that

$$X_t = x_0 + \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds, \quad t \geq 0,$$

   if and only if

$$X_t = x_0 + \int_0^t \sigma(X_s)\partial B_s + \int_0^t \tilde{b}(X_s)ds, \quad t \geq 0.$$

   where $\tilde{b} = b - \frac{1}{2}(\partial/\partial x)$.

4. Let $h : [0, \infty) \to \mathbb{R}$ be a measurable, square integrable function on $[0, t]$, for all $t \geq 0$. Show that the process $H_t = \int_0^t h(s)\, dB_s$ is Gaussian, and compute its covariances. (A real-valued process $(X_t, t \geq 0)$ is Gaussian if for any finite family $0 \leq t_1 < t_2 < \cdots < t_n < \infty$, the random vector $(X_{t_1}, \ldots, X_{t_n})$ is Gaussian.)

5. Let B be a standard Brownian motion. Use Itô's formula to prove that the following processes are martingales with respect to the natural filtration of B:

   (a) $X_t = \exp(\lambda^2 t/2)\sin(\lambda B_t)$

   (b) $X_t = (B_t + t)\exp(-B_t - t/2)$.

6. Let B be a standard Brownian motion and let $(\mathcal{F}_t)_{t\geq 0}$ be its natural filtration. Let $\mathcal{G}_t$ be the σ-algebra generated by $\mathcal{F}_t$ and B_1. Prove that for $0 \leq s \leq t \leq 1$,

$$\mathbb{E}\left[B_t - B_s|\mathcal{G}_s\right] = \frac{t-s}{1-s}(B_1 - B_s).$$

In particular, observe that B is *not* a $(\mathcal{G}_t)_{t\geq 0}$-martingale. Prove, however, that

$$\beta_t = B_t - \int_0^{t\wedge 1} \frac{B_1 - B_s}{1-s} ds$$

is a continuous $(\mathcal{G}_t)_{t\geq 0}$-martingale and so B is a continuous $(\mathcal{G}_t)_{t\geq 0}$-semimartingale. What is the quadratic variation of β? Conclude that β is a $(\mathcal{G}_t)_{t\geq 0}$-Brownian motion.
[Hint: Show $\mathbb{E}\left[(B_t - B_s)\,\mathbf{1}_{A\cap\{B_1\in D\}}\right] = \frac{t-s}{1-s}\mathbb{E}\left[(B_1 - B_s)\,\mathbf{1}_{A\cap\{B_1\in D\}}\right]$, $A \in \mathcal{F}_s$, $D \in \mathcal{B}(\mathbb{R})$.]
(This demonstrates that if a process is a continuous semimartingale in two different filtrations $(\mathcal{F}_t)_{t\geq 0}$ and $(\mathcal{G}_t)_{t\geq 0}$, its Doob-Meyer decompositions may be different, even when $\mathcal{F}_t \subseteq \mathcal{G}_t$ for all t.)

7. Let M be a continuous local martingale started from 0. Use Itô's formula to find the Doob-Meyer decomposition of $|M|^p$ when $p \geq 2$. Deduce, via Hölder's inequality, that there exists a constant $C_p \in (0, \infty)$ such that

$$\mathbb{E}\left[(M^*)^p\right] \leq C_p \mathbb{E}([M]_\infty^{p/2}),$$

where $M^* = \sup_{t\geq 0}|M_t|$.
(This is a special case of a *Burkholder-Davis-Gundy inequality*: for every $p \in (0, \infty)$, there exist constants c_p and C_p such that for all local martingales M with $M_0 = 0$,

$$c_p\mathbb{E}([M]_\infty^{p/2}) \leq \mathbb{E}\left[(M^*)^p\right] \leq C_p\mathbb{E}([M]_\infty^{p/2}).$$

See Revuz and Yor for a proof.)

8. (a) Let X be a positive continuous martingale such that $X_0 = x_0$ and $X_t \to 0$ a.s. as $t \to \infty$. Define $X^* = \sup_t X_t$. Let $x > 0$. Prove that

$$\mathbb{P}\left(X^* \geq x\right) = \min\{1, x_0/x\}$$

[Hint: Stop the martingale when it first exceeds x.]

(b) Let B be a standard Brownian motion. Show that $B_t - bt \to -\infty$ a.s. as $t \to \infty$, for all $b > 0$. Using part (a) and the fact that $\exp(aB_t - a^2t/2)$ is a martingale, show that the distribution of $\sup_{t\geq 0}(B_t - bt)$ is exponential with parameter $2b$.

9. Let N be a Poisson process of rate 1. Set $M_t = N_t - t$. Show that both M_t and $M_t^2 - t$ are martingales. Why does this not contradict Lévy's characterization of Brownian motion?

10. (a) Let $f : \mathbb{C} \to \mathbb{C}$ be analytic and let $Z_t = X_t + iY_t$ where (X, Y) is a Brownian motion in $\mathbb{R}^2$. Use Itô's formula to show that $M = f(Z)$ is a local martingale (in $\mathbb{R}^2$). Show further that M is a time-change of Brownian motion in $\mathbb{R}^2$.

(b) Let $D = \{|z| \leq 1\}$ and take $z \in D$. What is the hitting distribution for Z on ∂D in the case $Z_0 = 0$? By considering an analytic map $f : D \to D$ with $f(0) = z$, determine the hitting distribution for Z on ∂D when $Z_0 = z$.

For enthusiasts: a more complicated application of Itô's formula

This section contains two worked questions intended to introduce the important (but non-examinable) concept of the *local time* of a Brownian motion. This is a measure of how much "time" the process spends at a particular level (e.g. 0). Since the Lebesgue measure of the zero set of a Brownian motion is 0 (see Exercise 9.5.9 of Advanced Probability), this is not so straightforward! The first question (adapted from Øksendal) gives one way of defining the local time; the second shows that the local time increases only when the Brownian motion is at 0. For more information, see Revuz and Yor (Chapter VI) or Rogers and Williams (Vol 2, Section IV.43).

11. Let B be a standard Brownian motion. Suppose that we would like to apply Itô's formula to $f(B_t)$, where $f(x) = |x|$. f is not C^2 at $x = 0$ and so we need to modify it. Take $\epsilon > 0$ and set

$$f_\epsilon(x) = \begin{cases} |x| & \text{if } |x| \geq \epsilon \\ \frac{1}{2}(\epsilon + x^2/\epsilon) & \text{if } |x| < \epsilon. \end{cases}$$

Then f_ϵ is twice differentiable but f''_ϵ is discontinuous at $\pm\epsilon$.

(a) By choosing $g_n \in C^2(\mathbb{R})$ such that $g_n \to f_\epsilon$ uniformly, $g'_n \to f'_\epsilon$ unformly and $|g''_n| \leq M$, with $g''_n \to f''_\epsilon$ except at $x = \pm\epsilon$, show that

$$f_\epsilon(B_t) = f_\epsilon(B_0) + \int_0^t f'_\epsilon(B_s)dB_s + \frac{1}{2\epsilon}|\{s \in [0,t] : B_s \in (-\epsilon, \epsilon)\}|,$$

where $|A|$ denotes the Lebesgue measure of the set A.

(b) Prove that

$$\int_0^t f'_\epsilon(B_s)\mathbb{1}_{\{B_s \in (-\epsilon,\epsilon)\}}dB_s = \int_0^t \frac{B_s}{\epsilon}\mathbb{1}_{\{B_s \in (-\epsilon,\epsilon)\}}dB_s \to 0$$

in $L^2(\mathbb{P})$ as $\epsilon \to 0$.
[Hint: apply the Itô isometry to

$$\mathbb{E}\left[\left(\int_0^t \frac{B_s}{\epsilon}\mathbb{1}_{\{B_s \in (-\epsilon,\epsilon)\}}dB_s\right)^2\right]$$

to see this.]

(c) By letting $\epsilon \to 0$, prove that

$$|B_t| = |B_0| + \int_0^t \text{sgn}(B_s)dB_s + L_t, \qquad (*)$$

where

$$L_t = \lim_{\epsilon \to 0} \frac{1}{2\epsilon}|\{s \in [0,t] : B_s \in (-\epsilon, \epsilon)\}|$$

(with the limit in $L^2(\mathbb{P})$) and

$$\text{sgn}(x) = \begin{cases} -1 & \text{for } x \leq 0 \\ 1 & \text{for } x > 0. \end{cases}$$

L_t is called the local time of the Brownian motion at 0 and $(*)$ is *Tanaka's formula*.

12. Note that $(L_t)_{t\geq 0}$ must be increasing and continuous. So the expression $(*)$ tells us that $|B|$ is a semimartingale. Apply Itô's formula to both $|B|$ and B with $f(x) = x^2$ to obtain two different expressions for B_t^2. Comparing them, show that

$$\int_0^t |B_s| dL_s = 0 \text{ a.s.}$$

and deduce that L_t only increases on the set $\{t : B_t = 0\}$. Hence, L is measuring the "time" B spends at 0.