Stochastic Calculus and Applications **Lent 2007**

Example Sheet 1

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Finite variation functions

1. Let $a \in L^1[0,\infty)$ and set
$$x(t) = \int_0^t a(s)ds.$$
Show that x is continuous and of finite variation, with
$$v(t) = \int_0^t |a(s)|ds,$$
and that if h is non-negative and Borel measurable, then
$$\int_0^t h(s)\,dx(s) \;=\; \int_0^t h(s)\,a(s)\,ds.$$
[Hint: show that if one side above makes sense then so does the other and they are equal.]

2. Let $a : [0,\infty) \to \mathbb{R}$ and $x : [0,\infty) \to \mathbb{R}$ be continuous and suppose a is of finite variation. Show that, for all $t \geq 0$,
$$\lim_{n\to\infty} \sum_{k=0}^{\lfloor 2^n t\rfloor -1} (a((k+1)2^{-n}) - a(k2^{-n}))(x((k+1)2^{-n}) - x(k2^{-n})) = 0.$$

Previsible processes

3. Suppose B is a standard Brownian motion.
   (a) Let $T = \inf\{t \geq 0 : B_t = 1\}$. Show that H defined by $H_t = \mathbb{1}_{\{T\geq t\}}$ is a previsible process.
   (b) Let
$$\mathrm{sgn}(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases}$$
   Show that $(\mathrm{sgn}(B_t))_{t>0}$ is a previsible process (which is neither left- nor right-continuous).

4. Let H be a previsible process. Show that H_t is $\mathcal{F}_{t-}$-measurable for all $t > 0$, where $\mathcal{F}_{t-} = \sigma(\mathcal{F}_s : s < t)$.

5. Suppose that H and K are previsible processes which are bounded on compact time intervals (i.e. $\sup_{s\leq t} H_s < \infty$ and $\sup_{s\leq t} K_s < \infty$ for all t). Show that if A is a càdlàg adapted finite variation process then
$$H \cdot (K \cdot A) = (HK) \cdot A.$$
[Hint: use a monotone class argument.]

6. Let H be a left-continuous adapted process which is bounded on compact time intervals and let A be a càdlàg adapted finite variation process. Show that

$$(H \cdot A)_t = \lim_{n\to\infty} \sum_{k=0}^{\infty} H_{k2^{-n}} (A_{(k+1)2^{-n}\wedge t} - A_{k2^{-n}\wedge t})$$

with convergence uniform on compact time intervals. (Consider the limit ω by ω.)

Stopping times and martingales

7. Let X be an adapted process and $A \subseteq \mathbb{R}$. Define $T(\omega) = \inf\{t \geq 0 : X_t(\omega) \in A\}$.
   (a) If A is open, X is right-continuous and the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous (i.e. $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$), prove that T is a stopping time.
   (b) Prove that if S is a stopping time with respect to the natural filtration then the properties $S(\omega) \leq t$ and $X_s(\omega) = X_s(\omega')$ for all $s \leq t$ together imply that $S(\omega) = S(\omega')$.
   (c) Find an example where X is continuous but T is not a stopping time.

8. Let Y be a random variable taking values ± 1, each with probability $\frac{1}{2}$. Set $\mathcal{F}_t = \{\emptyset, \Omega\}$ for $t \leq 1$ and $\mathcal{F}_t = \sigma(Y)$ for $t > 1$. Show that all càdlàg $(\mathcal{F}_t)_{t\geq 0}$-martingales are constant. (So, in this case, the map $M \mapsto M_\infty : \mathcal{M}^2 \to L^2(\mathcal{F}_\infty)$ is not onto. When $(\mathcal{F}_t)_{t\geq 0}$ satisfies the *usual conditions* (i.e. completeness and right continuity) we know that, for all $X \in L^2(\mathcal{F}_\infty)$ the process $X_t = \mathbb{E}[X|\mathcal{F}_t]$ has a càdlàg version, which is an L^2-bounded martingale with $X_\infty = X$ a.s., so the map is onto.)

Local martingales and quadratic variation

9. Suppose that M is a bounded continuous martingale of finite variation. Prove that

$$M_t^2 = M_0^2 + 2\int_0^t M_s dM_s$$

and show that M_t^2 is a martingale. Use this to give another proof that $M \equiv M_0$ a.s.

10. Show that a uniform limit of càdlàg processes is càdlàg.

11. Let M be a continuous local martingale starting from 0. Show that M is an L^2-bounded martingale if and only if $\mathbb{E}([M]_\infty) < \infty$.

12. Show that covariation $[M, N]$ of continuous local martingales M and N is a symmetric bilinear form.

13. (a) Suppose that M and N are two independent continuous local martingales. Show that $[M, N]_t = 0$ for all $t \geq 0$. In particular, if $B^{(1)}$ and $B^{(2)}$ are the co-ordinates of a standard Brownian motion in $\mathbb{R}^2$, this shows that $[B^{(1)}, B^{(2)}]_t = 0$ for all $t \geq 0$.
   (b) Now let B be a standard Brownian motion in $\mathbb{R}$ and let T be a stopping time which is not constant a.s. By considering B^T and $B - B^T$, show that the converse to (a) is false. [Hint: show that T is measurable with respect to the σ-algebras generated by both B^T and $B - B^T$, using the fact that $[B^T]_t = t \wedge T$.]

14. Let X^n be a sequence of càdlàg processes and let T_k a sequence of random times, with $T_k \uparrow \infty$ a.s. as $k \to \infty$. Suppose $|||(X^n)^{T_k}||| \to 0$ as $n \to \infty$ for all k. Show that $X^n \to 0$ u.c.p.