Hand-out 2

Please send corrections to Stefan Grosskinsky: stefan@statslab.cam.ac.uk

Theorem 2.4. Quadratic variation
For each $M \in \mathcal{M}_{c,loc}$ there exists a (up to versions) unique continuous adapted increasing process $[M]$ such that $M^2 - [M] \in \mathcal{M}_{c,loc}$. Moreover, for $[M]^n_t := \sum_{k=0}^{\lfloor 2^n t\rfloor -1} \big(M_{(k+1)2^{-n}} - M_{k2^{-n}}\big)^2$
we have $[M]^n \to [M]$ $u.c.p.$ *as $n \to \infty$. We call $[M]$ the quadratic variation process of M.*

Lemma. Let $M \in \mathcal{M}$ be bounded. Suppose that $l \in \mathbb{N}$ and $0 = t_0 < t_1 < \ldots < t_l < \infty$. Then $\mathbb{E}\bigg(\Big(\sum_{k=0}^{l-1}(M_{t_{k+1}} - M_{t_k})^2\Big)^2\bigg)$ is bounded.

Proof of the Lemma. First note that

$$\mathbb{E}\bigg(\Big(\sum_{k=0}^{l-1}(M_{t_{k+1}}-M_{t_k})^2\Big)^2\bigg) = \sum_{k=0}^{l-1}\mathbb{E}\big((M_{t_{k+1}}-M_{t_k})^4\big)+2\sum_{k=0}^{l-1}\mathbb{E}\bigg((M_{t_{k+1}}-M_{t_k})^2\sum_{j=k+1}^{l-1}(M_{t_{j+1}}-M_{t_j})^2\bigg) \quad (1)$$

For each fixed k we have

$$\mathbb{E}\bigg((M_{t_{k+1}}-M_{t_k})^2\sum_{j=k+1}^{l-1}(M_{t_{j+1}}-M_{t_j})^2\bigg) = \mathbb{E}\bigg((M_{t_{k+1}}-M_{t_k})^2\mathbb{E}\Big(\sum_{j=k+1}^{l-1}(M_{t_{j+1}}-M_{t_j})^2\Big|\mathcal{F}_{t_{k+1}}\Big)\bigg)$$
$$= \mathbb{E}\bigg((M_{t_{k+1}} - M_{t_k})^2\mathbb{E}\Big(\sum_{j=k+1}^{l-1}(M^2_{t_{j+1}} - M^2_{t_j})\Big|\mathcal{F}_{t_{k+1}}\Big)\bigg) =$$
$$= \mathbb{E}\Big((M_{t_{k+1}} - M_{t_k})^2\mathbb{E}\big(M^2_{t_l} - M^2_{t_{k+1}}\big|\mathcal{F}_{t_{k+1}}\big)\Big) = \mathbb{E}\big((M_{t_{k+1}} - M_{t_k})^2(M^2_{t_l} - M^2_{t_{k+1}})^2\big) .$$

After inserting this in (1) we get the estimate

$$\mathbb{E}\bigg(\Big(\sum_{k=0}^{l-1}(M_{t_{k+1}}-M_{t_k})^2\Big)^2\bigg) \leq \mathbb{E}\bigg(\Big(\sup_j|M_{t_{j+1}}-M_{t_j}|^2 + 2\sup_j|M_{t_l}-M_{t_j}|^2\Big)\sum_{k=0}^{l-1}(M_{t_{k+1}}-M_{t_k})^2\bigg) .$$

Now, M is uniformly bounded by C, say. So using the inequality $(x-y)^2 \leq 2(x^2+y^2)$, we obtain

$$\mathbb{E}\bigg(\Big(\sum_{k=0}^{l-1}(M_{t_{k+1}}-M_{t_k})^2\Big)^2\bigg) \leq 12C^2\mathbb{E}\bigg(\sum_{k=0}^{l-1}(M_{t_{k+1}}-M_{t_k})^2\bigg) = 12C^2\mathbb{E}\big((M_{t_l}-M_{t_0})^2\big) \leq 48C^4 .$$

Proof of Theorem 2.4. Wlog we will consider the case $M_0 = 0$.
Uniqueness: If A and A' are two increasing processes satisfying the conditions for $[M]$ then

$$A_t - A'_t = (M^2_t - A'_t) - (M^2_t - A_t) \in \mathcal{M}_{c,loc}$$

is of finite variation and thus $A \equiv A'$ $a.s.$ by Theorem 1.10.

Existence: First we assume that M is bounded, which implies $M \in \mathcal{M}_c^2$. Fix $T > 0$ deterministic. Let

$$H_t^n = M_{2^{-n}\lfloor 2^n t\rfloor}^T = \sum_{k=0}^{\lfloor 2^n T\rfloor - 1} M_{k2^{-n}} \, \mathbb{1}_{(k2^{-n},(k+1)2^{-n}]}(t) \ .$$

Then $H^n \in \mathcal{S}$ for all $n \in \mathbb{N}$. Hence X^n defined by

$$X_t^n = (H^n \cdot M)_t = \sum_{k=0}^{\lfloor 2^n T\rfloor - 1} M_{k2^{-n}} \big(M_{(k+1)2^{-n}\wedge t} - M_{k2^{-n}\wedge t} \big) \ ,$$

is in $\mathcal{M}_c^2$ by Proposition 2.1 and by continuity of M. Recall that $\|X^n\| = \|X_\infty^n\|_2 = \|X_T\|_2$ since X_t^n is constant for $t \geq T$. For $n \geq m$ we have

$$\begin{aligned}
\|X^n - X^m\|^2 &= \mathbb{E}\big(((H^n - H^m)\cdot M)_T^2 \big) \leq \\
&\leq \mathbb{E}\bigg(\sup_{0\leq t\leq T} |H_t^n - H_t^m|^2 \sum_{k=0}^{\lfloor 2^n T\rfloor - 1} \big(M_{(k+1)2^{-n}} - M_{k2^{-n}}\big)^2 \bigg) \leq \\
&\leq \mathbb{E}\big(\sup_{0\leq t\leq T} |H_t^n - H_t^m|^4 \big)^{1/2} \mathbb{E}\bigg(\Big(\sum_{k=0}^{\lfloor 2^n T\rfloor - 1} \big(M_{(k+1)2^{-n}} - M_{k2^{-n}}\big)^2 \Big)^2 \bigg)^{1/2}
\end{aligned}$$

by Hölder's inequality. Since M is bounded, the second term is bounded by the lemma, and

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} |H_t^n - H_t^m|^4 \Big) = \mathbb{E}\Big(\sup_{0\leq t\leq T} |M_{2^{-n}\lfloor 2^n t\rfloor} - M_{2^{-m}\lfloor 2^m t\rfloor}|^4 \Big) \to 0 \quad \text{as } n, m \to \infty$$

by uniform continuity of M on $[0,T]$ and by bounded convergence. Hence X^n is a Cauchy sequence in $\big(\mathcal{M}_c^2, \|.\|\big)$ and so, by Proposition 2.3, converges to a limit $Y = (Y_t, 0 \leq t \leq T) \in \mathcal{M}_c^2$.
Now for any n and $1 \leq k \leq \lfloor 2^n T\rfloor$,

$$M_{k2^{-n}}^2 - 2X_{k2^{-n}}^n = \sum_{j=0}^{k-1} \big(M_{(j+1)2^{-n}} - M_{j2^{-n}}\big)^2 = [M]_{k2^{-n}}^n \ .$$

Hence, $M_t^2 - 2X_t^n$ is increasing along the sequence of times $\big(k2^{-n}, 1 \leq k \leq \lfloor 2^n T\rfloor\big)$. Passing to the limit $n \to \infty$, $M_t^2 - 2X_t^n$ must be $a.s.$ increasing. Set $[M]_t := M_t^2 - 2Y_t$ for $t \in [0,T]$ on the set where $M^2 - 2Y$ is increasing and $[M] \equiv 0$ otherwise. Hence, $[M]$ is a continuous increasing process and $M^2 - [M] = 2Y$ is a martingale on $[0,T]$.
We extend the definition of $[M]_t$ to $t \in [0,\infty)$ by applying the foregoing for all $T \in \mathbb{N}$. Note that the process $[M]$ obtained with T is the restriction to $[0,T]$ of $[M]$ defined with $T+1$.
Now, $\quad [M]_t^n = M_{2^{-n}\lfloor 2^n t\rfloor}^2 - 2\,X_{2^{-n}\lfloor 2^n t\rfloor}^n \quad$ and so

$$\sup_{0\leq t\leq T} \big|[M]_t - [M]_t^n\big| \leq \sup_{0\leq t\leq T} \big|M_{2^{-n}\lfloor 2^n t\rfloor}^2 - M_t^2\big| + 2 \sup_{0\leq t\leq T} \big|X_{2^{-n}\lfloor 2^n t\rfloor}^n - Y_{2^{-n}\lfloor 2^n t\rfloor}\big| + 2 \sup_{0\leq t\leq T} \big|Y_{2^{-n}\lfloor 2^n t\rfloor} - Y_t\big| \ .$$

Each term on the right converges to 0 in probability as $n \to \infty$, the first and third by continuity and the second because $X^n \to Y$ in $\big(\mathcal{M}_c^2, \|.\|\big)$ implies $\sup_{0\leq t\leq T} |X_t^n - Y_t| \to 0$ in L^2.
Now we turn to the general case $M \in \mathcal{M}_{c,loc}$. Define $T_n := \inf\big\{t \geq 0 \,:\, |M_t| \geq n\big\}$. Then $(T_n)_{n\in\mathbb{N}}$ reduces M and we can apply the bounded case to M^{T_n}, writing $A^n = [M^{T_n}]$. By uniqueness, $A_{t\wedge T_n}^{n+1}$ and A_t^n are indistinguishable, i.e. $\mathbb{P}\big(A_{t\wedge T_n}^{n+1} = A_t^n \text{ for all } t \geq 0\big) = 1$. Thus there exists an increasing process A such that for all $n \in \mathbb{N}$, $A_{t\wedge T_n}$ and A_t^n are indistinguishable. By construction, $(M_{t\wedge T_n}^2 - A_{t\wedge T_n})_{t\geq 0} \in \mathcal{M}_c$ and so $(M_t^2 - A_t)_{t\geq 0} \in \mathcal{M}_{c,loc}$ and we take $[M]_t = A_t$.
Finally, we have $\ [M^{T_n}]^m \to [M^{T_n}]\ u.c.p.$ as $m \to \infty$ for each $n \in \mathbb{N}$. Note that for each $t \geq 0$, $\mathbb{P}(t \leq T_n) \to 1$ as $n \to \infty$ to obtain that $\quad [M]^m \to [M] \quad u.c.p. \quad$ as $m \to \infty$. □