Stochastic Calculus and Applications — Lent 2007

www.statslab.cam.ac.uk/~stefan/teaching/stochcalc.html

Hand-out 1

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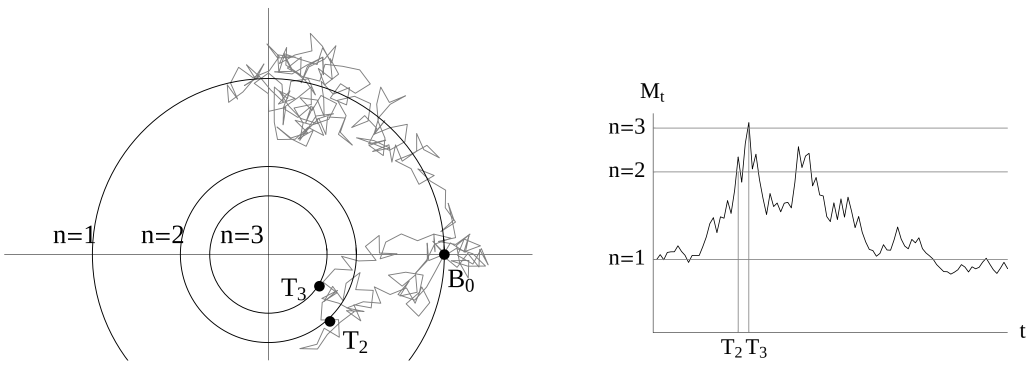
Example of a local martingale

Let B be a standard Brownian motion in $\mathbb{R}^3$ with $|B_0| = 1$, and let $M_t := 1/|B_t|$. In Ex. 9.5.8 of Adv. Prob. it was shown that $(M_t)_{t\geq 0}$ is a supermartingale with $\mathbb{E}(M_t) \to 0$ as $t \to \infty$, which is basically due to $|B_t| \to \infty$ $a.s.$ in $\mathbb{R}^3$. Since $\mathbb{E}(M_1) = 1$, M is not a martingale.

But M is a local martingale. To see this, define $(T_n)_{n\in\mathbb{N}}$ with

$$T_n := \inf\big\{t \geq 0 \,:\, |B_t| < 1/n\big\} = \inf\big\{t \geq 0 \,:\, |M_t| > n\big\} \,.$$



$(T_n)_{n\in\mathbb{N}}$ reduces M:

Recall from Proposition 6.4.3 of Adv. Prob. that if $f \in C^2(\mathbb{R}^3, \mathbb{R})$ with bounded derivatives, then

$$f(B_t) - f(B_0) - \frac{1}{2}\int_0^t \Delta f(B_s)\, ds \,, \quad t \geq 0$$

is a martingale. Take $f(x) = 1/|x|$, which is harmonic (i.e. $\Delta f = 0$) on $\big\{x \in \mathbb{R}^3 \,:\, |x| \geq 1/n\big\}$. Hence $(M_t^{T_n})_{t\geq 0}$ is a martingale for every $n \in \mathbb{N}$.

$T_n \nearrow \infty$ $a.s.$:

$T_1(\omega) = 0$ but for $n > 1$, $T_n(\omega)$ is non-trivial and, as $|B|$ cannot hit $1/(n+1)$ without having first hit $1/n$, $T_n(\omega)$ is increasing in n. Moreover, we know that Brownian motion in $\mathbb{R}^3$ does not hit single points. Indeed, almost every path $\big\{B_t(\omega) \,:\, t \geq 0\big\}$ is bounded away from 0. To see this, note that M^{T_n} is a bounded martingale. Set $S_R = \inf\big\{t \geq 0 \,:\, |B_t| > R\big\}$. Then by OST

$$\mathbb{E}(M_{T_n\wedge S_R}) = \mathbb{E}(M_0) = 1 \quad \text{and} \quad \mathbb{E}(M_{T_n\wedge S_R}) = n\mathbb{P}(T_n < S_R) + \frac{1}{R}\,\mathbb{P}(T_n \geq S_R) \,,$$

which implies for $n \geq 2$ that

$$\mathbb{P}(T_n < S_R) = \frac{1 - 1/R}{n - 1/R} \to \frac{1}{n} \quad \text{as } R \to \infty \,.$$

Since $\{T_n \not\to \infty\} = \{T_n < \infty \text{ for all } n \in \mathbb{N}\}$, this implies that $T_n \nearrow \infty$ $a.s.$. In particular $\mathbb{P}\big(\inf_{t\geq 0} |B_t| \leq 1/n\big) = 1/n$ and for large enough n (depending on the path) $T_n(\omega) = \infty$.