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www.statslab.cam.ac.uk/˜stefan/teaching/probmeas.html Part IID, M 2006

Probability and Measure

Example sheet 4

4.1 Let μ_1, μ_2 be finite measures on $(\mathbb{R}, \mathcal{B})$ such that $\int_{\mathbb{R}} g\, d\mu_1 = \int_{\mathbb{R}} g\, d\mu_2$ for all bounded continuous functions $g : \mathbb{R} \to \mathbb{R}$. Show that $\mu_1 = \mu_2$.

4.2 Let μ be a finite measure on $(\mathbb{R}, \mathcal{B})$ with Fourier transform $\hat{\mu}$. Show the following:

(a) $\hat{\mu}$ is a bounded continuous function.

(b) If $\int_{\mathbb{R}} |x|^k\, \mu(dx) < \infty$, then $\hat{\mu}$ has a k-th continuous derivative, which at 0 is given by

$$\hat{\mu}^{(k)}(0) = i^k \int_{\mathbb{R}} x^k\, \mu(dx) \ .$$

4.3 Let X be a real-valued random variable with characteristic function ϕ_X.

(a) Show that $\phi_X(u) \in \mathbb{R}$ for all $u \in \mathbb{R}$ if and only if $-X \sim X$, i.e. $\mu_{-X} = \mu_X$.

(b) Suppose that $|\phi_X(u)| = 1$ for all $|u| < \epsilon$ with some $\epsilon > 0$. Show that X is a.s. constant. (Hint: Take an independent copy X' of X, calculate $\phi_{X-X'}$ to see that $X = X'$ $a.s.$.)

4.4 By considering characteristic functions or otherwise, show that there do not exist iidrv's X, Y such that $X - Y$ is uniformly distributed on $[-1, 1]$.

4.5 The Cauchy distribution has density function $f(x) = \dfrac{1}{\pi(1 + x^2)}$, $x \in \mathbb{R}$.

(a) Show that the corresponding characteristic function is given by $\phi(u) = e^{-|u|}$.

(b) Show also that, if $X_1, \dots, X_n$ are independent Cauchy random variables, then $(X_1 + \dots + X_n)/n$ is also Cauchy.
Comment on this in the light of the strong law of large numbers and the central limit theorem.

4.6 Let $X, Y \sim \mathcal{N}(0, 1)$ and $Z \sim \mathcal{N}(0, \sigma^2)$ be independent Gaussian random variables. Calculate the characteristic function of $\eta = XY - Z$.

4.7 Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$ and $a, b \in \mathbb{R}$. Prove Proposition 5.3, i.e. show that

(a) $\mathbb{E}(X) = \mu$,

(b) $\mathrm{var}(X) = \sigma^2$,

(c) $aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$,

(d) $\phi_X(u) = e^{iu\mu - u^2\sigma^2/2}$.

4.8 Let $X_1, \ldots, X_n$ be independent $\mathcal{N}(0,1)$ random variables. Show that

$$\Big(\overline{X}, \sum_{m=1}^{n} (X_m - \overline{X})^2\Big) \quad \text{and} \quad \Big(X_n/\sqrt{n}, \sum_{m=1}^{n-1} X_m^2\Big)$$

have the same distribution, where $\overline{X} = (X_1 + \cdots + X_n)/n$.

4.9 Show that the shift map θ of Definition 6.3 is measurable and measure-preserving.

4.10 Let $E = [0,1)$ with Lebesgue measure. For $a \in E$ consider the mapping

$$\theta_a : E \to E \ , \quad \theta_a(x) = (x + a) \text{ mod } 1 \ .$$

(a) Show that θ_a is measure-preserving.
(b) Show that θ_a is not ergodic when a is rational.
(c) Show that θ_a is ergodic when a is irrational.
(Hint: Consider $a_n = \int_E f(x)\, e^{2\pi i n x}\, dx$ to show that every invariant function is constant.)
(d) Let $f : E \to \mathbb{R}$ be integrable. Determine for each $a \in E$ the limit function

$$\bar{f} = \lim_{n\to\infty} \big(f + f \circ \theta_a + \ldots + f \circ \theta_a^{n-1}\big)/n \ .$$

4.11 Show that $\theta(x) = 2x \mod 1$ is a measure-preserving transformation on $E = [0,1)$ with Lebesgue measure, and that θ is ergodic. Find $\bar{f}$ for each integrable function f.
(Hint: Consider the binary expansion $x = 0.x_1 x_2 x_3 \ldots$ and use that $X_n(x) = x_n$ are iidrvs with $\mathbb{P}(X_n = 0) = \mathbb{P}(X_n = 1) = \frac{1}{2}$, which is proved on hand-out 2.)

4.12 Call a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ on a common probability space *stationary* if for each $n, k \in \mathbb{N}$ the random vectors $(X_1, \ldots, X_n)$ and $(X_{k+1}, \ldots, X_{k+n})$ have the same distribution, i.e. for $A_1, \ldots, A_n \in \mathcal{B}$,

$$\mathbb{P}(X_1 \in A_1, \ldots, X_n \in A_n) = \mathbb{P}(X_{k+1} \in A_1, \ldots, X_{k+n} \in A_n) \ .$$

Show that, if $(X_n)_{n\in\mathbb{N}}$ is a stationary sequence and $X_1 \in L^p$, for some $p \in [1, \infty)$, then

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to X \ \ a.s. \text{ and in } L^p \ ,$$

for some random variable $X \in L^p$, and find $\mathbb{E}(X)$.

4.13 Find a sequence $(X_n)_{n\in\mathbb{N}}$ of independent random variables with $\mathbb{E}\big(|X_n|\big) < \infty$ and $\mathbb{E}(X_n) = 0$ for all $n \in \mathbb{N}$, such that $(X_1 + \ldots + X_n)/n$ does not almost surely converge to 0.

4.14 Let $(X_n)_{n\in\mathbb{N}}$ be independent random variables with $\mathbb{P}(X_n = 0) = \mathbb{P}(X_n = 1) = \frac{1}{2}$, and define

$$U_n = X_1 X_2 + X_2 X_3 + \ldots + X_{2n} X_{2n+1} \ .$$

Show that $U_n/n \to c \ \ a.s.$ for some $c \in \mathbb{R}$, and determine c.