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www.statslab.cam.ac.uk/~stefan/teaching/probmeas.html

Part IID, M 2006

Probability and Measure

Example sheet 3

Unless otherwise specified, let $(E, \mathcal{E}, \mu)$ be a measure space and $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space.

3.1 Let μ be the Lebesgue measure on $\big(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)\big)$.

(a) For $f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$ calculate the iterated Lebesgue integrals

$\int_0^1 \int_0^1 f(x,y)\,dx\,dy$ and $\int_0^1 \int_0^1 f(x,y)\,dy\,dx$.

What does the result tell about the double integral $\int_{(0,1)^2} f\,d\mu$?

(b) Show that for $f(x,y) = \begin{cases} \frac{x\,y}{(x^2+y^2)^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$ the iterated integrals

$\int_{-1}^1 \int_{-1}^1 f(x,y)\,dx\,dy$ and $\int_{-1}^1 \int_{-1}^1 f(x,y)\,dy\,dx$

coincide, but that the double integral $\int_{(-1,1)^2} f\,d\mu$ does not exist.

(c) Let ν be the counting measure on $(\mathbb{R}, \mathcal{B})$, i.e. $\nu(A)$ is equal to the number of elements in A whenever A is finite, and $\nu(A) = \infty$ otherwise. Denote by $\Delta = \big\{(x,y) \in (0,1)^2 \ : \ x = y\big\}$ the diagonal in $(0,1)^2$ and calculate the iterated integrals

$\int_0^1 \int_0^1 \mathbb{1}_\Delta(x,y)\,dx\,\nu(dy)$ and $\int_0^1 \int_0^1 \mathbb{1}_\Delta(x,y)\,\nu(dy)\,dx$.

Does the result contradict Fubini's theorem?

3.2 (a) Are the following statements equivalent? (Justify your answer.)

(i) f is continuous almost everywhere, (ii) $f = g$ $a.e.$ for a continuous function g.

(b) Let $X_n \sim U\big([-1/n, 1/n]\big)$ be uniform random variables on $[-1/n, 1/n]$ for $n \in \mathbb{N}$. Do the X_n converge, and if yes in what sense?

3.3 Prove that the space $L^\infty(E, \mathcal{E}, \mu)$ is complete.

3.4 Let $p \in [1, \infty]$ and let $f_n, f \in L^p(E, \mathcal{E}, \mu)$ for $n \in \mathbb{N}$. Show that:

$f_n \to f$ in L^p $\Rightarrow$ $f_n \to f$ in measure , but the converse is not true .

3.5 Read hand-out 2 carefully. Find examples which show that the reverse implications, concerning the concepts of convergence on page 1, are in general false. How does the picture change if the measure space $(\Omega, \mathcal{A}, \mathbb{P})$ is not finite?

3.6 Let X be a random variable in $\mathbb{R}$ and let $1 \leq p < q < \infty$. Show that

$$\mathbb{E}\big(|X|^p\big) = \int_0^\infty p\,\lambda^{p-1}\,\mathbb{P}\big(|X| \geq \lambda\big)\,d\lambda$$

and deduce: $X \in L^q(\mathbb{P}) \ \Rightarrow \ \mathbb{P}\big(|X| \geq \lambda\big) = O(\lambda^{-q}) \ \Rightarrow \ X \in L^p(\mathbb{P})$.

Remark on questions 3.7(a) and 3.8(a): Start with an indicator function and extend your argument to the general case, analogous to the proof of Lemma 3.14(ii).

3.7 A *stepfunction* $g : \mathbb{R} \to \mathbb{R}$ is any finite linear combination of indicator functions of finite intervals.

(a) Show that the set of stepfunctions $\mathcal{I}$ is dense in $L^p(\mathbb{R})$ for all $p \in [1, \infty)$,
i.e. for all $f \in L^p(\mathbb{R})$ and every $\epsilon > 0$ there exists $g \in \mathcal{I}$ such that $\|f - g\|_p < \epsilon$.
(Hint: Use the result of question 1.9.)

(b) Using (a), argue that the set of continuous functions $C(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, $p \in [1, \infty)$.

3.8 (a) Show that, if X and Y are independent random variables, then $\|X\,Y\|_1 = \|X\|_1\,\|Y\|_1$, but that the converse is in general not true.

(b) Show that, if X and Y are independent and integrable, then $\mathbb{E}(X\,Y) = \mathbb{E}(X)\,\mathbb{E}(Y)$.

3.9 Let $V_1 \subseteq V_2 \subseteq \ldots$ be an increasing sequence of closed subspaces of $L^2 = L^2(E, \mathcal{E}, \mu)$.
For $f \in L^2$, denote by f_n the orthogonal projection of f on V_n. Show that f_n converges in L^2.

3.10 Given a countable family of disjoint events $(G_i)_{i \in I}$, $G_i \in \mathcal{A}$, with $\bigcup_{i \in I} G_i = \Omega$.
Set $\mathcal{G} = \sigma\big(G_n : n \in \mathbb{N}\big)$ and $V = L^2(\Omega, \mathcal{G}, \mathbb{P})$.
Show that, for $X \in L^2(\Omega, \mathcal{A}, \mathbb{P})$, the conditional expectation $\mathbb{E}(X \,|\, \mathcal{G})$ is a version of the orthogonal projection of X on V.

3.11 (a) Find a sequence of random variables $(X_n)_{n \in \mathbb{N}}$ which is not bounded in L^1, but satisfies the other condition for uniform integrability, i.e.

$$\forall\, \epsilon > 0 \;\; \exists\, \delta > 0 \;\; \forall\, A \in \mathcal{A} \;\; \forall\, i \in I \;:\; \mathbb{P}(A) < \delta \;\Rightarrow\; \mathbb{E}\big(|X_i|\, \mathbb{1}_A\big) < \epsilon \,.$$

(b) Find a uniformly integrable sequence of random variables $(X_n)_{n \in \mathbb{N}}$ such that

$$X_n \to 0 \;\; a.s. \quad \text{and} \quad \mathbb{E}\big(\sup_n |X_n|\big) = \infty \,.$$

3.12 Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of identically distributed r.v.s in $L^2(\mathbb{P})$. Show that, as $n \to \infty$,

(a) for all $\epsilon > 0$, $\quad n\,\mathbb{P}\big(|X_1| > \epsilon\sqrt{n}\big) \to 0$,

(b) $n^{-1/2} \max_{k \leq n} |X_k| \to 0$ in probability ,

(c) $n^{-1/2} \max_{k \leq n} |X_k| \to 0$ in L^1 .

3.13 The *moment generating function* M_X of a real-valued random variable X is defined by

$$M_X(\theta) = \mathbb{E}\big(e^{\theta X}\big)\,, \quad \theta \in \mathbb{R}.$$

(a) Show that the maximal domain of definition $I = \big\{\theta \in \mathbb{R} \,:\, M_X(\theta) < \infty\big\}$ is an interval and find examples for $I = \mathbb{R}$, $\{0\}$ and $(-\infty, 1)$.

Assume for simplicity that $X \geq 0$ from now on.

(b) Show that if I contains a neighbourhood of 0 then X has finite moments of all orders given by $\mathbb{E}(X^n) = \Big(\frac{d}{d\theta}\Big)^n\Big|_{\theta=0} M_X(\theta)$.

(c) Find a necessary and sufficient condition on the sequence of moments $m_n = \mathbb{E}(X^n)$ for I to contain a neighbourhood of 0.