Stefan Grosskinsky
www.statslab.cam.ac.uk/∼stefan/teaching/probmeas.html

Part IID, M 2006

Probability and Measure

Example sheet 2

Unless otherwise specified, let $(E,\mathcal{E})$, $(F,\mathcal{F})$ be measurable spaces and $(\Omega,\mathcal{A},\mathbb{P})$ be a probability space.

2.1 Let $f : E \to F$ be any function (not necessarily measurable).

(a) Show that $f^{-1}\big(\sigma(\mathcal{A})\big) = \sigma\big(f^{-1}(\mathcal{A})\big)$ for all $\mathcal{A} \subseteq \mathcal{P}(F)$.

(b) Let f be $\mathcal{E}/\mathcal{F}$-measurable. Under which circumstances is $f(\mathcal{E}) \subseteq \mathcal{P}(F)$ a σ-algebra?

(c) Take $(E,\mathcal{E}) = (F,\mathcal{F}) = (\mathbb{R},\mathcal{B})$. Find the σ-algebras $\sigma(f_i)$ generated by the functions $f_1(x) = x\ ,\quad f_2(x) = x^2\ ,\quad f_3(x) = |x|\ ,\quad f_4(x) = \mathbb{1}_{\mathbb{Q}}(x)\ .$

2.2 Let $f_n : E \to \overline{\mathbb{R}}$, $n \in \mathbb{N}$ be $\mathcal{E}/\overline{\mathcal{B}}$-measurable functions. Show that also the following functions are measurable, whenever they are well defined:

(a) $f_1 + f_2$ (b) $\inf_{n\in\mathbb{N}} f_n$ (c) $\sup_{n\in\mathbb{N}} f_n$ (d) $\liminf_{n\to\infty} f_n$ (e) $\limsup_{n\to\infty} f_n$.

(f) Deduce further that: $\big\{x \in E\ :\ f_n(x) \text{ converges as } n \to \infty\big\} \in \mathcal{E}$

2.3 Let $f : E \to \mathbb{R}^d$ be written in the form $f(x) = \big(f_1(x),\ldots,f_d(x)\big)$. Show that f is measurable w.r.t. $\mathcal{E}$ and $\mathcal{B}(\mathbb{R}^d)$ if and only if each $f_i : E \to \mathbb{R}$ is measurable w.r.t. $\mathcal{E}$ and $\mathcal{B}$.

2.4 Skorohod representation theorem
Let $F_n : \mathbb{R} \to [0,1]$, $n \in \mathbb{N}$ be probability distribution functions. Consider the probability space $(\Omega,\mathcal{A},\mathbb{P})$ where $\Omega = (0,1]$, $\mathcal{A} = \mathcal{B}((0,1])$ are the Borel sets on $(0,1]$ and $\mathbb{P}$ is the restriction of Lebesgue measure to $\mathcal{A}$. For each n define $X_n : (0,1] \to \mathbb{R}\ ,\quad X_n(\omega) = \inf\big\{x\ :\ \omega \le F_n(x)\big\}$.

(a) Show that the X_n are random variables with distributions F_n. Are the X_n independent?

(b)* Suppose $F(x)$ is a probability distribution function such that $\lim_{n\to\infty} F_n(x) = F(x)$ for all $x \in \mathbb{R}$ at which F is continuous. Let $X : (0,1] \to \mathbb{R}$ be a random variable with distribution F defined analogously to the X_n. Show that $X_n \to X$ $a.s.$.

2.5 Let $X_1, X_2, \ldots$ be random variables on $(\Omega,\mathcal{A},\mathbb{P})$.

(a) Show that X_1 and X_2 are independent if and only if

$$\mathbb{P}\big(X_1 \le x,\ X_2 \le y\big) = \mathbb{P}(X_1 \le x)\,\mathbb{P}(X_2 \le y) \quad \text{for all } x,y \in \mathbb{R}\ . \qquad (*)$$

(b) Suppose $(*)$ holds for all pairs X_i, X_j, $i \ne j$. Is this sufficient for the $(X_n)_{n\in\mathbb{N}}$ to be independent? Justify your answer.

(c) Let X_1, X_2 be independent and identically distributed. Show that $X_1 = X_2$ almost surely implies that X_1 and X_2 are almost surely constant.

2.6 Let $X_1, X_2, \ldots$ be random variables with $X_n \xrightarrow{D} X$. Show that then also $h(X_n) \xrightarrow{D} h(X)$ for all continuous functions $h : \mathbb{R} \to \mathbb{R}$. (Hint: Use the Skorohod representation theorem)

2.7 Let $X_1, X_2, \ldots$ be random variables on $(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathcal{T}$ the tail σ-algebra of $(X_n)_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$ let $S_n = X_1 + \ldots + X_n$. Which of the following are tail events in $\mathcal{T}$,

$$\{X_n \le 0\ ev.\}\,, \quad \{S_n \le 0\ i.o.\}\,, \quad \big\{\liminf_{n\to\infty} S_n \le 0\big\}\,, \quad \big\{\lim_{n\to\infty} S_n \text{ exists}\big\}\ ?$$

2.8 Let $X_1, X_2, \ldots$ be random variables with $\quad X_n = \begin{cases} n^2 - 1 & \text{with probability } 1/n^2 \\ -1 & \text{with probability } 1 - 1/n^2 \end{cases}$.

Show that $\mathbb{E}\Big(\dfrac{X_1 + \cdots + X_n}{n}\Big) = 0$ for each n , but $\dfrac{X_1 + \cdots + X_n}{n} \to -1$ almost surely.

2.9 Let $X, X_1, X_2, \ldots$ be random variables on $(\Omega, \mathcal{A}, \mathbb{P})$.

(a) Show that $\big\{\omega \in \Omega \,:\, X_n(\omega) \to X(\omega)\big\} \in \mathcal{A}$.

(b) Show that $X_n \to X$ almost surely $\Leftrightarrow$ $\sup_{m\ge n} |X_m - X| \to 0$ in probability .

2.10 Let $X_1, X_2, \ldots$ be independent random variables with distribution $\mathcal{N}(0,1)$. Prove that

$$\limsup_{n\to\infty} \big(X_n/\sqrt{2\log n}\big) = 1 \qquad \text{a.s.}$$

(Hint: Consider the events $A_n = \big\{X_n > \alpha\sqrt{2\log n}\big\}$ for $\alpha \in (0,\infty)$.)

2.11 Show that, as $n \to \infty$,

(a) $\displaystyle\int_0^\infty \sin(e^x)/\big(1 + nx^2\big)\, dx \to 0$, $\qquad$ (b) $\displaystyle\int_0^1 (n\cos x)/\big(1 + n^2 x^{3/2}\big)\, dx \to 0$.

2.12 Let $u,\ v : \mathbb{R} \to \mathbb{R}$ be differentiable on $[a,b]$ with continuous derivatives u' and v'. Show that for $a < b$

$$\int_a^b u(x)\, v'(x)\, dx = \big[u(b)\, v(b) - u(a)\, v(a)\big] - \int_a^b u'(x)\, v(x)\, dx\ .$$

2.13 Let $\phi : [a,b] \to \mathbb{R}$ be continuously differentiable and strictly increasing. Show that for all continuous functions g on $\big[\phi(a), \phi(b)\big]$

$$\int_{\phi(a)}^{\phi(b)} g(y)\, dy = \int_a^b g\big(\phi(x)\big)\, \phi'(x)\, dx\ .$$

2.14 Show that the function $\quad f(x) = x^{-1}\sin x \quad$ is not Lebesgue integrable over $[1,\infty)$ but that

$$\lim_{y\to\infty} \int_0^y f(x)\, dx = \frac{\pi}{2}\ . \qquad \text{(use e.g. Fubini's theorem and } x^{-1} = \textstyle\int_0^\infty e^{-xt}\, dt\text{)}$$

2.15 (a) Let μ be a measure on $(E, \mathcal{E})$ and $f : E \to [0,\infty)$ be $\mathcal{E}/\mathcal{B}$-measurable with $\int_E f\, d\mu < \infty$. Define $\nu(A) = \int_A f\, d\mu$ for each $A \in \mathcal{E}$. Show that ν is a measure on $(E, \mathcal{E})$ and that

$$\int_E g\, d\nu = \int_E f\, g\, d\mu \quad \text{for all integrable } g : E \to \mathbb{R}\ .$$

(b) Let μ be a σ-finite measure on $(E, \mathcal{E})$. Show that for all $\mathcal{E}/\mathcal{B}$-measurable $g : E \to [0,\infty)$

$$\int_E g\, d\mu = \int_0^\infty \mu(g \ge \lambda)\, d\lambda\ .$$