Stefan Grosskinsky
www.statslab.cam.ac.uk/~stefan/teaching/probmeas.html

Part IID, M 2006

Probability and Measure

Example sheet 1

1.1 Let E be a set.

(a) Let $\mathcal{F} \subseteq \mathcal{P}(E)$ be the set of all finite sets and their complements (called *cofinite* sets). Show that $\mathcal{F}$ is an algebra.

(b) Let $\mathcal{G} \subseteq \mathcal{P}(E)$ be the set of all countable sets and their complements. Show that $\mathcal{G}$ is a σ-algebra.

(c) Give a simple example of E and σ-algebras $\mathcal{E}_1$, $\mathcal{E}_2$, such that $\mathcal{E}_1 \cup \mathcal{E}_2$ is not a σ-algebra.

1.2 A non-empty set A in a σ-algebra $\mathcal{E}$ is called an *atom*, if there is no proper subset $B \subseteq A$ such that $B \in \mathcal{E}$. Let $A_1, \ldots, A_N$ be non-emtpy subsets of a set E.

(a) If the A_n are mutually disjoint and $\bigcup_n A_n = E$, how many elements does $\sigma\big(\{A_1, \ldots, A_N\}\big)$ have and what are its atoms?

(b) Show that in general $\sigma\big(\{A_1, \ldots, A_N\}\big)$ consists of only finitely many sets.

1.3 Show that the following families of subsets of $\mathbb{R}$ generate the same σ-algebra $\mathcal{B}$:
(i) $\{(a,b) : a < b\}$, (ii) $\{(a,b] : a < b\}$, (iii) $\{(-\infty, b] : b \in \mathbb{R}\}$.

1.4 A σ-algebra is called *separable* if it can be generated by a countable family of sets. Show that the Borel σ-algebra $\mathcal{B}$ of $\mathbb{R}$ is separable.

1.5 For which σ-algebras on $\mathbb{R}$ are the following set-functions measures:

$$\mu_1(A) = \begin{cases} 0 \ , \ \text{if } A = \emptyset \\ 1 \ , \ \text{if } A \neq \emptyset \end{cases} , \qquad \mu_2(A) = \begin{cases} 0 \ , \ \text{if } A = \emptyset \\ \infty \ , \ \text{if } A \neq \emptyset \end{cases} , \qquad \mu_3(A) = \begin{cases} 0 \ , \ \text{if } A \text{ is finite} \\ 1 \ , \ \text{if } A^c \text{ is finite} \end{cases} ?$$

1.6 Let $\mathcal{E}$ be a ring on E and $\mu : \mathcal{E} \to [0, \infty]$ an additive set function. Show that:

(a) If μ is continuous from below at all $A \in \mathcal{E}$ it is also countably additive.

(b) If μ is countably additive it is also countably **sub**additive.

1.7 Let $(E, \mathcal{E}, \mu)$ be a measure space. Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of sets in $\mathcal{E}$, and define

$$\liminf A_n = \bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} A_m \ , \qquad \limsup A_n = \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} A_m \ .$$

(a) Show that $\mu(\liminf A_n) \leq \liminf_{n \to \infty} \mu(A_n)$.

(b) Show that $\mu(\limsup A_n) \geq \limsup_{n \to \infty} \mu(A_n)$ if $\mu(E) < \infty$.
Give an example with $\mu(E) = \infty$ when this inequality fails.

1.8 (a) Show that a π-system which is also a d-system is a σ-algebra.

(b) Give an example of a d-system that is not a σ-algebra.

1.9 (a) Find a Borel set that cannot be written as a countable union of intervals.

(b) Let $B \in \mathcal{B}$ be a Borel set with $\lambda(B) < \infty$, where λ is the Lebesgue measure. Show that, for every $\epsilon > 0$, there exists a finite union of disjoint intervals $A = (a_1, b_1] \cup \ldots \cup (a_n, b_n]$ such that $\lambda(A \triangle B) < \epsilon$, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$.
(Hint: First consider all bounded sets B for which the conclusion holds and show that they form a d-system.)

1.10 Completion

Let $(E, \mathcal{E}, \mu)$ be a measure space. A subset N of E is called *null* if $N \subseteq B$ for some $B \in \mathcal{E}$ with $\mu(B) = 0$. Write $\mathcal{N}$ for the set of all null sets.

(a) Prove that the family of subsets $\mathcal{C} = \big\{A \cup N \,:\, A \in \mathcal{E},\ N \in \mathcal{N}\big\}$ is a σ-algebra.

(b) Show that the measure μ may be extended to a measure μ' on $\mathcal{C}$ with $\mu'(A \cup N) = \mu(A)$.

The σ-algebra $\mathcal{C}$ is called the *completion* of $\mathcal{E}$ with respect to μ.

1.11 Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of events in the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, i.e. $A_n \in \mathcal{A}$ for all n. Show that the A_n, $n \in \mathbb{N}$, are independent if and only if the σ-algebras which they generate, $\mathcal{A}_n = \{\emptyset, A_n, A_n^c, \Omega\}$, are independent.

1.12 (a) Let μ_F be the Lebesgue-Stieltjes measure on $\mathbb{R}$ associated with the distribution function F. Show that F is continuous at x if and only if $\mu_F\big(\{x\}\big) = 0$.

(b) Let $(F_n)_{n\in\mathbb{N}}$, be a sequence of distribution functions on $\mathbb{R}$ such that $F(x) = \lim\limits_{n\to\infty} F_n(x)$ exists for all $x \in \mathbb{R}$. Show that F need not be a distribution function.

1.13 Cantor set

Let $C_0 = [0, 1]$, and let $C_1, C_2, \ldots$ be constructed iteratively by deletion of middle-thirds.
Thus $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ and so on.
The set $C = \lim\limits_{n\to\infty} C_n = \bigcap\limits_{n\in\mathbb{N}} C_n$ is called the *Cantor set*.
Let F_n be the distribution function of the uniform probability measure concentrated on C_n.

(a) Show that C is uncountable and has Lebesgue measure 0.

(b) Show that the limit $F(x) = \lim\limits_{n\to\infty} F_n(x)$ exists for all $x \in [0, 1]$.
(Hint: Establish a recursion relation for $F_n(x)$ and use the contraction mapping theorem.)

(c) Show that F is continuous on $[0, 1]$ with $F(0) = 0$, $F(1) = 1$.

(d) Show that F is differentiable except on a set of measure 0, and that $F'(x) = 0$ wherever F is differentiable.

1.14 Riemann zeta function

The *Riemann zeta function* is given by $\zeta(s) = \sum\limits_{n=1}^{\infty} n^{-s}$, $s > 1$.

Let $s{>}1$ and $\mathbb{P}_f : \mathcal{P}(\mathbb{N}) \to [0, 1]$ be the probability measure with mass function $f(n) = n^{-s}/\zeta(s)$.
For $p \in \{1, 2, \ldots\}$ let $A_p = \{n \in \mathbb{N} \,:\, p \mid n\}$ (p divides n).

(a) Show that the events $\{A_p \,:\, p \text{ prime}\}$ are independent. Deduce Euler's formula

$$\frac{1}{\zeta(s)} = \prod_{p \text{ prime}} \Big(1 - \frac{1}{p^s}\Big) .$$

(b) Show that $\mathbb{P}_f\big(\{n \in \mathbb{N} \,:\, n \text{ is square-free}\}\big) = \dfrac{1}{\zeta(2s)}$.