

Probability and Measure

Hand-out 4

Ergodic Theorems

Let (E, \mathcal{E}, μ) be a σ -finite measure space with a meas.-preserving transformation $\theta : E \rightarrow E$.
 Let $f : E \rightarrow \mathbb{R}$ be integrable and define $S_n : E \rightarrow \mathbb{R}$ by $S_0 = 0$ and

$$S_n = S_n(f) = f + f \circ \theta + \dots + f \circ \theta^{n-1} \quad \text{for } n \geq 1.$$

Maximal ergodic lemma

Let $S^* = \sup_{n \in \mathbb{N}} S_n : E \rightarrow \mathbb{R}$. Then $\int_{S^* > 0} f d\mu \geq 0$.

Proof. Set $S_n^* = \max_{0 \leq m \leq n} S_m$ and $A_n = \{S_n^* > 0\}$. Then, for $m = 1, \dots, n$,

$$S_m = f + S_{m-1} \circ \theta \leq f + S_n^* \circ \theta.$$

On A_n we have $S_n^* = \max_{1 \leq m \leq n} S_m$, so $S_n^* \leq f + S_n^* \circ \theta$.

On A_n^c we have $S_n^* = 0 \leq S_n^* \circ \theta$. So, integrating and adding, we obtain

$$\int_E S_n^* d\mu \leq \int_{A_n} f d\mu + \int_E S_n^* \circ \theta d\mu.$$

But S_n^* is integrable and θ is measure-preserving, so

$$\int_E S_n^* \circ \theta d\mu = \int_E S_n^* d\mu < \infty \quad \text{which implies} \quad \int_{A_n} f d\mu \geq 0.$$

As $n \rightarrow \infty$, $A_n \nearrow \{S^* > 0\}$ so, by monotone convergence, $\int_{\{S^* > 0\}} f d\mu \geq 0$. □

Birkhoff's almost everywhere ergodic theorem

There exists $\bar{f} : E \rightarrow \mathbb{R}$ invariant, with $\int_E |\bar{f}| d\mu \leq \int_E |f| d\mu$ and $\frac{S_n}{n} \rightarrow \bar{f}$ a.e. as $n \rightarrow \infty$.

Proof. The functions $\liminf_n (S_n/n)$ and $\limsup_n (S_n/n)$ are invariant, since

$$\left(\liminf \frac{S_n}{n} \right) \circ \theta = \liminf \left(\frac{S_n \circ \theta}{n} \right) = \liminf \left(\frac{S_{n+1} - f}{n} \right) = \liminf \left(\frac{S_{n+1}}{n+1} \right).$$

Therefore, for $a < b$,

$$D = D(a, b) = \left\{ \liminf_n (S_n/n) < a < b < \limsup_n (S_n/n) \right\}.$$

is an invariant event.

We shall show that $\mu(D) = 0$. First, by invariance, we can restrict everything to D and thereby reduce to the case $D = E$. Note that either $b > 0$ or $a < 0$. We can interchange the two cases by replacing f by $-f$. Let us assume then that $b > 0$.

Let $B \in \mathcal{E}$ with $\mu(B) < \infty$, then $g = f - b \mathbb{1}_B$ is integrable and, for each $x \in D$, for some n ,

$$S_n(g)(x) \geq S_n(f)(x) - nb > 0.$$

Hence $S^*(g) > 0$ everywhere and, by the maximal ergodic lemma,

$$0 \leq \int_D (f - b \mathbb{1}_B) d\mu = \int_D f d\mu - b\mu(B).$$

Since μ is σ -finite, we can let $B \nearrow D$ to obtain $b\mu(D) \leq \int_D f d\mu$.

In particular, we see that $\mu(D) < \infty$. A similar argument applied to $-f$ and $-a$, this time with $B = D$, shows that $(-a)\mu(D) \leq \int_D (-f) d\mu$. Hence $b\mu(D) \leq \int_D f d\mu \leq a\mu(D)$.

Since $a < b$ and the integral is finite, this forces $\mu(D) = 0$.

Back to general E . Set

$$\Delta = \left\{ \liminf_n (S_n/n) < \limsup_n (S_n/n) \right\},$$

then Δ is invariant. Also, $\Delta = \bigcup_{a,b \in \mathbb{Q}, a < b} D(a,b)$, so $\mu(\Delta) = 0$. On the complement of Δ , S_n/n converges in $[-\infty, \infty]$, so we can define an invariant function $\bar{f} : E \rightarrow \overline{\mathbb{R}}$ by

$$\bar{f} = \begin{cases} \lim_n (S_n/n) & \text{on } \Delta^c \\ 0 & \text{on } \Delta \end{cases}.$$

Finally, we have $\int_E |f \circ \theta^n| d\mu = \int_E |f| d\mu$, so $\int_E |S_n| d\mu \leq n \int_E |f| d\mu$ for all n . Hence, by Fatou's lemma,

$$\int_E |\bar{f}| d\mu = \int_E \liminf_n |S_n/n| d\mu \leq \liminf_n \int_E |S_n/n| d\mu \leq \int_E |f| d\mu. \quad \square$$

Von Neumann's L^p ergodic theorem

Assume that $\mu(E) < \infty$ and $p \in [1, \infty)$. Then, for $f \in L^p$, $S_n/n \rightarrow \bar{f}$ (from Birkhoff) in L^p .

Proof. Since θ is measure-preserving we have

$$\|f \circ \theta^n\|_p = \|f \circ \theta^{n-1}\|_p = \dots = \|f\|_p.$$

So, by Minkowski's inequality, $\|S_n(f)/n\|_p \leq \|f\|_p$.

Given $\epsilon > 0$, choose $K < \infty$ so that $\|f - g\|_p < \epsilon/3$, where $g = (-K) \vee f \wedge K$.

By Birkhoff's theorem, $S_n(g)/n \rightarrow \bar{g}$ a.e.. We have $|S_n(g)/n| \leq K$ for all n so, by bounded convergence, there exists N such that, for $n \geq N$,

$$\|S_n(g)/n - \bar{g}\|_p < \epsilon/3.$$

By Fatou's lemma,

$$\|\bar{f} - \bar{g}\|_p^p = \int_E \liminf_n \left| \frac{S_n(f-g)}{n} \right|^p d\mu \leq \liminf_n \int_E \left| \frac{S_n(f-g)}{n} \right|^p d\mu \leq \|f - g\|_p^p.$$

Hence, for $n \geq N$,

$$\left\| \frac{S_n(f)}{n} - \bar{f} \right\|_p \leq \left\| \frac{S_n(f-g)}{n} \right\|_p + \left\| \frac{S_n(g)}{n} - \bar{g} \right\|_p + \|\bar{g} - \bar{f}\|_p < 3\epsilon/3 = \epsilon. \quad \square$$