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www.statslab.cam.ac.uk/∼stefan/teaching/probmeas.html Part IID, M 2005

Probability and Measure

Hand-out 3

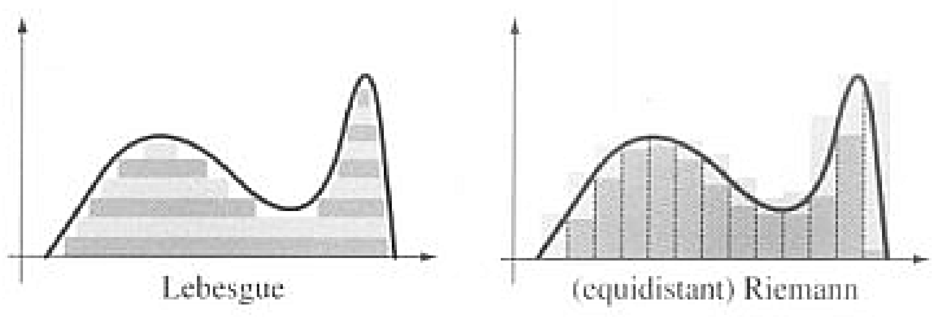
Connection between Lebesgue and Riemann integration

Definition. $f : [a, b] \to \mathbb{R}$ is *Riemann integrable* (R-integrable) with integral $R \in \mathbb{R}$, if

$$\forall\, \epsilon > 0 \;\; \exists\, \delta > 0 \;\; \forall\, x_j \in I_j \;:\; \Big| R - \sum_{j=1}^{n} f(x_j)\,|I_j| \Big| < \epsilon\,, \tag{1}$$

for some finite partition $\{I_1, \ldots, I_n\}$ of $[a, b]$ into subintervals of lengths $|I_j| < \delta$.

This corresponds to an approximation of f by step functions $\sum_{j=1}^{n} f(x_j)\, \mathbb{1}_{I_j}$, a special case of simple functions which are constant on intervals.



The picture is taken from R.L. Schilling, *Measures, Integrals and Martingales*, CUP 2005. He writes:

> ... the Riemann sums partition the domain of the function without taking into account the shape of the function, thus slicing up the area under the function *vertically*. Lebesgue's approach is exactly the opposit: the domain is partitioned according to the values of the function at hand, leading to a *horizontal* decomposition of the area.

Theorem. Lebesgue's integrability criterium
$f : [a, b] \to \mathbb{R}$ is R-integrable if and only if f is bounded on $[a, b]$ and continuous almost everywhere, i.e. the set of points in $[a, b]$ where f is not continuous has Lebesgue measure 0.

Corollary. *Suppose $f : [a, b] \to \mathbb{R}$ is R-integrable. Then it is also Lebesgue integrable (L-integrable) and the values of both integrals coincide.*

Proof. For a partition $\{I_1, \ldots, I_n\}$ define the step functions

$$\bar{g}_n = \sum_{j=1}^{n} \sup\{f(x) : x \in I_j\} \, \mathbb{1}_{I_j} \ , \quad \underline{g}_n = \sum_{j=1}^{n} \inf\{f(x) : x \in I_j\} \, \mathbb{1}_{I_j} \ .$$

Thus $\underline{g}_n \leq f \leq \bar{g}_n$ and if f is continuous $a.e.$, $\underline{g}_n, \bar{g}_n \to f \ \ a.e.$ as $n \to \infty$ and $|I_j| \to 0$.
Since f is bounded, it follows by dominated convergence for the L-integrals

$$\int_a^b \underline{g}_n(x)\,dx \to \int_a^b f(x)\,dx \ , \quad \int_a^b \bar{g}_n(x)\,dx \to \int_a^b f(x)\,dx \ ,$$

so the sum in (1) converges and $\quad R = \int_a^b f(x)\,dx$.
On the other hand, if the sum in (1) converges, then $\int_a^b \big|\bar{g}_n(x) - \underline{g}_n(x)\big|\,dx \to 0$ and thus for the limit functions $\underline{g} = \bar{g} \ \ a.e.$, and f is continuous $a.e.$ since $\underline{g} \leq f \leq \bar{g}$.
If f was not bounded, one could choose x_j in (1) such that the sum does not converge. □

On the other hand, not every L-integrable function is also R-integrable. The standard example is $f = \mathbb{1}_{[0,1]\cap\mathbb{Q}}$, which can be made R-integrable by changing it on a set of L-measure 0.
This might suggest that for every L-integrable f there exists an R-integrable g with $f = g \ \ a.e.$.

This is not true as demonstrated by the following example:
Let $\{r_1, r_2 \ldots\}$ be an enumeration of the rationals in $(0,1)$. For small $\epsilon > 0$ and each $n \in \mathbb{N}$ choose an open interval $I_n \subseteq (0,1)$ with $r_n \in I_n$ and L-measure $\mu(I_n) < \epsilon\, 2^{-n}$. Put $A = \bigcup_n I_n$. Then A is dense in $(0,1)$ with $0 < \mu(A) < \epsilon$ and thus for any non-degenerate subinterval I of $(0,1)$, $\mu(A \cap I) > 0$.
Take $f = \mathbb{1}_A$ and suppose that $f = g$ a.e.. Let $\{I_j\}$ be some decomposition of $(0,1)$ into subintervals. Since for each j, $\mu\big(I_j \cap A \cap \{f = g\}\big) = \mu(I_j \cap A) > 0$, $g(x_j) = f(x_j) = 1$ for some $x_j \in I_j \cap A$, and thus

$$\sum_{j=1}^{n} g(x_j)\,\mu(I_j) = 1 > \mu(A) \ . \tag{2}$$

If g were R-integrable, its integral would have to coincide with the L-integral $\int_0^1 f\,d\mu = \mu(A)$, which is in contradiction to (2).

Product measure and Fubini's theorem

Let $(E_1, \mathcal{E}_1, \mu_1)$ and $(E_2, \mathcal{E}_2, \mu_2)$ be σ-finite measure spaces and $E = E_1 \times E_2$.

Theorem 3.15. Product measure
There exists a unique measure $\mu = \mu_1 \otimes \mu_2$ *on* $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2 = \sigma\big(\{A_1 \times A_2 : A_1 \in \mathcal{E}_1, A_2 \in \mathcal{E}_2\}\big)$,

such that $\quad \mu(A_1 \times A_2) = \mu_1(A_1)\,\mu_2(A_2) \quad$ *for all* $A_1 \in \mathcal{E}_1$ *and* $A_2 \in \mathcal{E}_2$,

defined as $\quad \mu(A) := \int_{E_1} \Big(\int_{E_2} \mathbb{1}_A(x_1, x_2)\,\mu_2(dx_2) \Big)\,\mu_1(dx_1)$.

Theorem 3.16. Fubini's theorem

(i) Let $f : E \to [0, \infty]$ *be* $\mathcal{E}$*-measurable. Then* $\quad \int_E f\,d\mu = \int_{E_1} \Big(\int_{E_2} f(x_1, x_2)\,\mu_2(dx_2) \Big)\,\mu_1(dx_1)$.

(ii) Let $f : E \to \overline{\mathbb{R}}$ *be* μ*-integrable. Then* $f(x_1, .)$ *is* μ_2*-integrable for* μ_1*-almost all* x_1.
$\int_{E_2} f(., x_2)\,\mu_2(dx_2) \quad$ *is* μ_1*-integrable and the formula in (i) holds.*