Stefan Grosskinsky
www.statslab.cam.ac.uk/∼stefan/teaching/probmeas.html Part IID, M 2006

Probability and Measure

Hand-out 1

Carathéodory's extension theorem
Let $\mathcal{E}$ be a ring on E and $\mu : \mathcal{E} \to [0,\infty]$ be a countably additive set function. Then there exists a measure μ' on $\big(E, \sigma(\mathcal{E})\big)$ such that $\mu'(A) = \mu(A)$ for all $A \in \mathcal{E}$.

Proof.
For any $B \subseteq E$, define the *outer measure* $\quad \mu^*(B) = \inf \sum_n \mu(A_n)$,

where the infimum is taken over all sequences $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that $B \subseteq \bigcup_n A_n$ and is taken to be ∞ if there is no such sequence. Note that μ^* is increasing and $\mu^*(\emptyset) = 0$. Let us say that $A \subseteq E$ is μ^*-*measurable* if, for all $B \subseteq E$,

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c) \ .$$

Write $\mathcal{M}$ for the set of all μ^*-measurable sets. *We shall show that $\mathcal{M}$ is a σ-algebra containing $\mathcal{E}$ and that μ^* is a measure on $\mathcal{M}$, extending μ.* This will prove the theorem.

Step I. We show that μ^* is countably subadditive.
Suppose that $B \subseteq \bigcup_n B_n$. If $\mu^*(B_n) < \infty$ for all n, then, given $\epsilon > 0$, there exist sequences $(A_{nm})_{m\in\mathbb{N}}$ in $\mathcal{E}$, with

$$B_n \subseteq \bigcup_m A_{nm}, \quad \mu^*(B_n) + \epsilon/2^n \geq \sum_m \mu(A_{nm}) \ .$$

Then $\quad B \subseteq \bigcup_n \bigcup_m A_{nm} \quad$ and thus $\quad \mu^*(B) \leq \sum_n \sum_m \mu(A_{nm}) \leq \sum_n \mu^*(B_n) + \epsilon$.

Hence, in any case $\quad \mu^*(B) \leq \sum_n \mu^*(B_n)$.

Step II. We show that μ^* extends μ.
Since $\mathcal{E}$ is a ring and μ is countably additive, μ is countably subadditive. Hence, for $A \in \mathcal{E}$ and any sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{E}$ with $A \subseteq \bigcup_n A_n$, we have $\quad \mu(A) \leq \sum_n \mu(A_n)$.

On taking the infimum over all such sequences, we see that $\mu(A) \leq \mu^*(A)$. On the other hand, it is obvious that $\mu^*(A) \leq \mu(A)$ for $A \in \mathcal{E}$.

Step III. We show that $\mathcal{M}$ contains $\mathcal{E}$.
Let $A \in \mathcal{E}$ and $B \subseteq E$. We have to show that

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c) \ .$$

By subadditivity of μ^*, it is enough to show that

$$\mu^*(B) \geq \mu^*(B \cap A) + \mu^*(B \cap A^c) \ .$$

If $\mu^*(B) = \infty$, this is trivial, so let us assume that $\mu^*(B) < \infty$. Then, given $\epsilon > 0$, we can find a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that

$$B \subseteq \bigcup_n A_n \;, \quad \mu^*(B) + \epsilon \geq \sum_n \mu(A_n) \;.$$

Then $\quad B \cap A \subseteq \bigcup_n (A_n \cap A) \;, \quad B \cap A^c \subseteq \bigcup_n (A_n \cap A^c) \;,$ so that

$$\mu^*(B \cap A) + \mu^*(B \cap A^c) \leq \sum_n \mu(A_n \cap A) + \sum_n \mu(A_n \cap A^c) = \sum_n \mu(A_n) \leq \mu^*(B) + \epsilon \;.$$

Since $\epsilon > 0$ was arbitrary, we are done.

Step IV. We show that $\mathcal{M}$ is an algebra.
Clearly $E \in \mathcal{M}$ and $A^c \in \mathcal{E}$ whenever $A \in \mathcal{E}$. Suppose that $A_1, A_2 \in \mathcal{M}$ and $B \subseteq E$. Then

$$\begin{aligned}
\mu^*(B) &= \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c) \\
&= \mu^*(B \cap A_1 \cap A_2) + \mu^*(B \cap A_1 \cap A_2^c) + \mu^*(B \cap A_1^c) \\
&= \mu^*(B \cap A_1 \cap A_2) + \mu^*(B \cap (A_1 \cap A_2)^c \cap A_1) + \mu^*(B \cap (A_1 \cap A_2)^c \cap A_1^c) \\
&= \mu^*(B \cap (A_1 \cap A_2)) + \mu^*(B \cap (A_1 \cap A_2)^c) \;.
\end{aligned}$$

Hence $A_1 \cap A_2 \in \mathcal{M}$.

Step V. We show that $\mathcal{M}$ is a σ-algebra and that μ^* is a measure on $\mathcal{M}$.
We already know that $\mathcal{M}$ is an algebra, so it suffices to show that, for any sequence of disjoint sets $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$, for $A = \bigcup_n A_n$ we have

$$A \in \mathcal{M} \;, \quad \mu^*(A) = \sum_n \mu^*(A_n).$$

So, take any $B \subseteq E$, then

$$\begin{aligned}
\mu^*(B) &= \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c) = \mu^*(B \cap A_1) + \mu^*(B \cap A_2) + \mu^*(B \cap A_1^c \cap A_2^c) \\
&= \ldots = \sum_{i=1}^{n} \mu^*(B \cap A_i) + \mu^*(B \cap A_1^c \cap \ldots \cap A_n^c) \;.
\end{aligned}$$

Note that $\mu^*(B \cap A_1^c \cap \ldots \cap A_n^c) \geq \mu^*(B \cap A^c)$ for all n. Hence, on letting $n \to \infty$ and using countable subadditivity, we get

$$\mu^*(B) \geq \sum_{n=1}^{\infty} \mu^*(B \cap A_n) + \mu^*(B \cap A^c) \geq \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

The reverse inequality holds by subadditivity, so we have equality. Hence $A \in \mathcal{M}$ and, setting $B = A$, we get $\quad \mu^*(A) = \sum_{n=1}^{\infty} \mu^*(A_n) \;.$ □