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www.statslab.cam.ac.uk/∼stefan/teaching/probmeas.html Part IID, M 2005

Probability and Measure

Example sheet 4

4.1 Let μ_1, μ_2 be finite measures on $(\mathbb{R},\mathcal{B})$ such that $\displaystyle\int_{\mathbb{R}} g\,d\mu_1 = \int_{\mathbb{R}} g\,d\mu_2$ for all bounded continuous functions $g:\mathbb{R}\to\mathbb{R}$. Show that $\mu_1=\mu_2$.

4.2 Show that the Fourier transform of a finite measure on $(\mathbb{R},\mathcal{B})$ is a bounded continuous function.

4.3 Let μ be a finite measure on $(\mathbb{R},\mathcal{B})$. Suppose the Fourier transform $\hat{\mu}$ is Lebesgue integrable. Show that μ has a continuous density function f with respect to Lebesgue measure, i.e.

$$\mu(A) = \int_A f(x)\,dx \quad \text{for all } A\ .$$

4.4 By considering characteristic functions or otherwise, show that there do not exist iidrv's X, Y such that $X - Y$ is uniformly distributed on $[-1,1]$.

4.5 The Cauchy distribution has density function $f(x) = \frac{1}{\pi(1+x^2)}\ ,\quad x\in\mathbb{R}$.

(a) Show that the corresponding characteristic function is given by $\phi(u) = e^{-|u|}$.
(b) Show also that, if $X_1,\ldots,X_n$ are independent Cauchy random variables, then $(X_1+\cdots+X_n)/n$ is also Cauchy.
Comment on this in the light of the strong law of large numbers and the central limit theorem.

4.6 For a finite measure μ on $(\mathbb{R},\mathcal{B})$ show that, if $\displaystyle\int_{\mathbb{R}} |x|^k\,\mu(dx) < \infty$, then the Fourier transform $\hat{\mu}$ of μ has a k-th continuous derivative, which at 0 is given by

$$\hat{\mu}^{(k)}(0) = i^k \int_{\mathbb{R}} x^k\,\mu(dx)\ .$$

4.7 Suppose $X\sim\mathcal{N}(\mu,\sigma^2)$ and $a,b\in\mathbb{R}$. Show that

(a) $\mathbb{E}(X)=\mu$,
(b) $\mathrm{var}(X)=\sigma^2$,
(c) $aX+b\sim\mathcal{N}(a\mu+b,a^2\sigma^2)$,
(d) $\phi_X(u)=e^{iu\mu-u^2\sigma^2/2}$.

4.8 Let $X_1,\ldots,X_n$ be independent $\mathcal{N}(0,1)$ random variables. Show that

$$\Big(\overline{X},\sum_{m=1}^n (X_m-\overline{X})^2\Big) \quad\text{and}\quad \Big(X_n/\sqrt{n},\sum_{m=1}^{n-1} X_m^2\Big)$$

have the same distribution, where $\overline{X}=(X_1+\cdots+X_n)/n$.

4.9 Suppose that $X_1, \ldots, X_n$ are jointly Gaussian random variables with

$$\mathbb{E}(X_i) = \mu_i \ , \quad \text{cov}(X_i, X_j) = \Sigma_{ij} \ ,$$

and that the matrix $\Sigma = (\Sigma_{ij})_{i,j=1,..,n}$ is invertible.

(a) Set $Y = \Sigma^{-1/2}(X - \mu)$. Show that $Y_1, \ldots, Y_n$ are independent $\mathcal{N}(0,1)$ random variables.

(b) Show that we can write X_2 in the form $X_2 = a\,X_1 + Z$ where Z is independent of X_1, and determine the distribution of Z.

4.10 Show that the shift map θ (of Definition 7.3) is measurable and measure-preserving.

4.11 Let $E = [0,1)$ with Lebesgue measure. For $a \in E$ consider the mapping

$$\theta_a : E \to E \ , \quad \theta_a(x) = (x + a) \bmod 1 \ .$$

(a) Show that θ_a is measure-preserving.

(b) Show that θ_a is not ergodic when a is rational.

(c) Show that θ_a is ergodic when a is irrational.
[Consider $a_n = \int_E f(x)\, e^{2\pi i n x}\, dx$ to show that every invariant function is constant.]

(d) Let $f : E \to \mathbb{R}$ be integrable. Determine for each $a \in E$ the limit function

$$\bar{f} = \lim_{n\to\infty} \big(f + f \circ \theta_a + \ldots + f \circ \theta_a^{n-1} \big)/n \ .$$

4.12 Show that $\theta(x) = 2x \mod 1$ is a measure-preserving transformation on $E = [0,1)$ with Lebesgue measure, and that θ is ergodic. Find $\bar{f}$ for each integrable function f.

4.13 Call a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ on a common probability space *stationary* if for each $n, k \in \mathbb{N}$ the random vectors $(X_1, \ldots, X_n)$ and $(X_{k+1}, \ldots, X_{k+n})$ have the same distribution, i.e. for $A_1, \ldots, A_n \in \mathcal{B}$,

$$\mathbb{P}(X_1 \in A_1, \ldots, X_n \in A_n) = \mathbb{P}(X_{k+1} \in A_1, \ldots, X_{k+n} \in A_n) \ .$$

Show that, if $(X_n)_{n\in\mathbb{N}}$ is a stationary sequence and $X_1 \in L^p$, for some $p \in [1, \infty)$, then

$$\frac{1}{n} \sum_{i=1}^{n} X_i \to X \ \ a.s. \text{ and in } L^p \ ,$$

for some random variable $X \in L^p$, and find $\mathbb{E}(X)$.

4.14 Find a sequence $(X_n)_{n\in\mathbb{N}}$ of independent random variables with $\mathbb{E}\big(|X_n|\big) < \infty$ and $\mathbb{E}(X_n) = 0$ for all $n \in \mathbb{N}$, such that $(X_1 + \ldots + X_n)/n$ does not almost surely converge to 0.

4.15 Let $f : (0, \infty) \to \mathbb{R}$ be a bounded continuous function, having Laplace transform

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda x} f(x)\, dx \ , \quad \lambda \in (0, \infty) \ .$$

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent exponential rv's of parameter λ. Show that $\hat{f}$ has derivatives of all orders on $(0, \infty)$ and that, for some $C(\lambda, n) \neq 0$ independent of f, we have

$$(d/d\lambda)^{n-1} \hat{f}(\lambda) = C(\lambda, n)\, \mathbb{E}\big(f(S_n)\big) \quad \text{for all } n \in \mathbb{N} \ ,$$

where $S_n = X_1 + \ldots + X_n$. Deduce that if $\hat{f} \equiv 0$ then also $f \equiv 0$.