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www.statslab.cam.ac.uk/∼stefan/teaching/probmeas.html

Part IID, M 2005

Probability and Measure

Example sheet 3

Unless otherwise specified, let $(E, \mathcal{E}, \mu)$ be a measure space and $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space.

3.1 Prove that the space $L^\infty(E, \mathcal{E}, \mu)$ is complete.

3.2 Let $p \in [1, \infty]$ and let $f_n, f \in L^p(E, \mathcal{E}, \mu)$ for $n \in \mathbb{N}$. Show that:

$$f_n \to f \text{ in } L^p \quad \Rightarrow \quad f_n \to f \text{ in measure }, \quad \text{but the converse is not true .}$$

3.3 Read hand-out 2 carefully. Find examples which show that the reverse implications, concerning the concepts of convergence on page 1, are in general false.

3.4 Let X be a random variable in $\mathbb{R}$ and let $1 \le p < q < \infty$. Show that

$$\mathbb{E}\big(|X|^p\big) = \int_0^\infty p\, \lambda^{p-1}\, \mathbb{P}\big(|X| \ge \lambda\big)\, d\lambda$$

and deduce: $\quad X \in L^q(\mathbb{P}) \;\Rightarrow\; \mathbb{P}\big(|X| \ge \lambda\big) = O(\lambda^{-q}) \;\Rightarrow\; X \in L^p(\mathbb{P})$.

Remark on questions 3.5 to 3.7, part (a): Start with an indicator function and extend your argument to the general case, analogous to the proof of Lemma 4.10(ii).

3.5 A *stepfunction* $g : \mathbb{R} \to \mathbb{R}$ is any finite linear combination of indicator functions of finite intervals.

(a) Show that the set of stepfunctions $\mathcal{I}$ is dense in $L^p(\mathbb{R})$ for all $p \in [1, \infty)$, i.e. for all $f \in L^p(\mathbb{R})$ and every $\epsilon > 0$ there exists $g \in \mathcal{I}$ such that $\|f - g\|_p < \epsilon$. [Use the result of question 1.9.]

(b) Using (a), argue that the set of continuous functions $C(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, $p \in [1, \infty)$.

3.6 Let $f \in L^1(E, \mathcal{E}, \mu)$ be non-negative and define $\quad \nu(A) := \displaystyle\int_E f\, \mathbb{1}_A\, d\mu$.

(a) Show that ν is a measure on $(E, \mathcal{E})$ and that $\quad \displaystyle\int_E g\, d\nu = \int_E g f\, d\mu$, for all measurable functions $g : E \to [0, \infty]$.

(b) Let X be a random variable in $\mathbb{R}$ with measurable *density function* $f_X : \mathbb{R} \to [0, \infty]$, i.e.

$$\mathbb{P}(X \in A) = \int_A f_X(x)\, dx \quad \text{for all } A \in \mathcal{B} \,.$$

Use (a) to show that $\quad \mathbb{E}\big(g(X)\big) = \displaystyle\int_{\mathbb{R}} g(x)\, f(x)\, dx \quad$ for all measurable $g : \mathbb{R} \to [0, \infty]$.

3.7 (a) Show that, if X and Y are independent random variables, then $\|X\,Y\|_1 = \|X\|_1\,\|Y\|_1$, but that the converse is in general not true.

(b) Show that, if X and Y are independent and integrable, then $\mathbb{E}(X\,Y) = \mathbb{E}(X)\,\mathbb{E}(Y)$.

3.8 Let $(X_n)_{n\in\mathbb{N}}$ be iid random variables with moment generating function $\phi(\theta) = \mathbb{E}\big(e^{\theta\,X_1}\big)$ and expected value m. Assume that ϕ is finite on some neighbourhood of the origin.

(a) Show that $\mathbb{P}(X > x) \leq \Lambda(x)$ for all $x > m$, where $\Lambda(x) = \inf\big\{e^{-\theta x}\,\phi(\theta)\;:\;\theta\in\mathbb{R}\big\}$.

(b) Show that $S_n = \sum_{i=1}^{n} X_i$ satisfies $\mathbb{P}(S_n > n\,x) \leq \Lambda(x)^n$ for all $x > m$.

(c) Compute $\Lambda(x)$ for $x > 1$, when the X_n have exponential distribution with parameter 1.

3.9 For $f, g \in L^2$ show Pythagoras' rule and the parallelogramm law,

$$\|f+g\|_2^2 = \|f\|_2^2 + 2\langle f,g\rangle + \|f\|_2^2\,, \qquad \|f+g\|_2^2 + \|f-g\|_2^2 = 2\big(\|f\|_2^2 + \|g\|_2^2\big)\,.$$

3.10 Show that every covariance matrix is non-negative definite.

3.11 Let $V_1 \subseteq V_2 \subseteq \ldots$ be an increasing sequence of closed subspaces of $L^2 = L^2(E,\mathcal{E},\mu)$. For $f\in L^2$, denote by f_n the orthogonal projection of f on V_n. Show that f_n converges in L^2.

3.12 Given a countable family of disjoint events $(G_i)_{i\in I}$, $G_i\in\mathcal{A}$, with $\bigcup_{i\in I} G_i = \Omega$. Set $\mathcal{G} = \sigma\big(G_n\;:\;n\in\mathbb{N}\big)$ and $V = L^2(\Omega,\mathcal{G},\mathbb{P})$. Show that, for $X\in L^2(\Omega,\mathcal{A},\mathbb{P})$, the conditional expectation $\mathbb{E}(X\,|\,\mathcal{G})$ is a version of the orthogonal projection of X on V.

3.13 (a) Find a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ which is not bounded in L^1, but satisfies the other condition for uniform integrability, i.e.

$$\forall\,\epsilon > 0\;\exists\,\delta > 0\;\forall\,A\in\mathcal{A}\;\forall\,i\in I\;:\;\mathbb{P}(A) < \delta\;\Rightarrow\;\mathbb{E}\big(|X_i|\,\mathbb{1}_A\big) < \epsilon\,.$$

(b) Find a uniformly integrable sequence of random variables $(X_n)_{n\in\mathbb{N}}$ such that

$$X_n \to 0\;\; a.s. \quad\text{and}\quad \mathbb{E}\big(\sup_n |X_n|\big) = \infty\,.$$

3.14 Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of iid random variables in $L^2(\mathbb{P})$. Show that, as $n\to\infty$,

(a) for all $\epsilon > 0$, $n\,\mathbb{P}\big(|X_1| > \epsilon\sqrt{n}\big) \to 0$,

(b) $n^{-1/2}\max_{k\leq n}|X_k| \to 0$ in probability,

(c) $n^{-1/2}\max_{k\leq n}|X_k| \to 0$ in L^1.