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Part IID, M 2005

Probability and Measure

Example sheet 2

Unless otherwise specified, let $(E, \mathcal{E})$ be a measurable space and $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space.

2.1 Let $f_n : E \to \mathbb{R}$, $n \in \mathbb{N}$ be $\mathcal{E}/\mathcal{B}$-measurable functions. Show that also the following functions, with values in $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$, are measurable:

(a) $f_1 + f_2$ (b) $\inf\limits_{n\in\mathbb{N}} f_n$ (c) $\sup\limits_{n\in\mathbb{N}} f_n$ (d) $\liminf f_n$ (e) $\limsup f_n$

(f) Deduce further that: $\big\{x \in E : f_n(x) \text{ converges as } n \to \infty\big\} \in \mathcal{E}$

2.2 Let $f : E \to \mathbb{R}^d$ be written in the form $f(x) = \big(f_1(x), \ldots, f_d(x)\big)$. Show that f is measurable w.r.t. $\mathcal{E}$ and $\mathcal{B}(\mathbb{R}^d)$ if and only if each $f_i : E \to \mathbb{R}$ is measurable w.r.t. $\mathcal{E}$ and $\mathcal{B}$.

2.3 Skorohod representation theorem
Let F, F_n, $n \in \mathbb{N}$ be probability distribution functions such that $F(x) = \lim_{n\to\infty} F_n(x)$ for all $x \in \mathbb{R}$ at which F is continuous. Show that there exists a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ together with random variables $X,\ X_n : \Omega \to \mathbb{R}$ such that:

(a) X has distribution function F ,

(b) X_n has distribution function F_n ,

(c) $X_n \to X$ almost surely as $n \to \infty$.

2.4 Let $X,\ Y$ be random variables on $(\Omega, \mathcal{A}, \mathbb{P})$. Show that X and Y are independent if and only if

$$\mathbb{P}\big(X \le x,\ Y \le y\big) = \mathbb{P}(X \le x)\,\mathbb{P}(Y \le y) \quad \text{for all } x, y \in \mathbb{R}\ .$$

2.5 Let $X_1, X_2, \ldots$ be random variables with $X_n = \begin{cases} n^2 - 1 & \text{with probability } 1/n^2 \\ -1 & \text{with probability } 1 - 1/n^2 \end{cases}$.

Show that $\mathbb{E}\Big(\dfrac{X_1 + \cdots + X_n}{n}\Big) = 0$ but $\dfrac{X_1 + \cdots + X_n}{n} \to -1$ almost surely.

2.6 Let $X_1, X_2, \ldots$ be independent random variables with distribution uniform on $[0, 1]$. Let A_n be the event that a record occurs at time n, that is,

$$A_n = \big\{\omega \in \Omega : X_n(\omega) > X_m(\omega) \quad \text{for all } m < n\big\}\ .$$

(a) Find the probability of A_n and show that $A_1, A_2, \ldots$ are independent.

(b) Deduce that, with probability one, infinitely many records occur.

2.7 Let $X_1, X_2, \dots$ be independent random variables with distribution $\mathcal{N}(0,1)$. Prove that

$$\limsup_n \left(X_n/\sqrt{2\log n}\right) = 1 \qquad \text{a.s.}$$

2.8 Let X be a non-negative integer-valued random variable.

(a) Show that $\quad \mathbb{E}(X) = \sum_{n=1}^{\infty} \mathbb{P}(X \geq n).$

(b) Deduce that, if $\mathbb{E}(X) = \infty$ and $X_1, X_2, \dots$ are iid random variables with the same distribution as X, $\quad \limsup(X_n/n) \geq 1$ a.s. and indeed $\quad \limsup(X_n/n) = \infty$ a.s.

(c) Now suppose that $Y_1, Y_2, \dots$ is any sequence of iid random variables with $\mathbb{E}|Y_1| = \infty$. Show that $\quad \limsup(|Y_n|/n) = \infty$ a.s. and indeed $\quad \limsup(|Y_1 + \cdots + Y_n|/n) = \infty$ a.s.

2.9 Show that, as $n \to \infty$,

(a) $\displaystyle\int_0^\infty \sin(e^x)/\left(1 + nx^2\right) dx \to 0\,,$ $\qquad$ (b) $\displaystyle\int_0^1 (n\cos x)/\left(1 + n^2 x^{3/2}\right) dx \to 0\,.$

2.10 Show that the function $\quad f(x) = x^{-1}\sin x \quad$ is not Lebesgue integrable over $[1,\infty)$ but that

$$\lim_{y\to\infty} \int_0^y f(x)\,dx = \frac{\pi}{2}\,. \qquad \text{(use e.g. Fubini's theorem and } x^{-1} = \textstyle\int_0^\infty e^{-xt}\,dt\text{)}$$

2.11 Let $u,\ v : \mathbb{R} \to \mathbb{R}$ be differentiable on $[a,b]$ with continuous derivatives u' and v'.
Show that for $a < b$

$$\int_a^b u(x)\,v'(x)\,dx = \big[u(b)\,v(b) - u(a)\,v(a)\big] - \int_a^b u'(x)\,v(x)\,dx\,.$$

2.12 Let $\phi : [a,b] \to \mathbb{R}$ be continuously differentiable and strictly increasing. Show that for all continuous functions g on $\big[\phi(a), \phi(b)\big]$

$$\int_{\phi(a)}^{\phi(b)} g(y)\,dy = \int_a^b g\big(\phi(x)\big)\,\phi'(x)\,dx\,.$$

2.13 Let μ be a σ-finite measure on $(E,\mathcal{E})$ and let $f : E \to [0,\infty)$ be $\mathcal{E}/\mathcal{B}$-measurable. Show that

$$\mu(f) = \int_0^\infty \mu(f \geq \lambda)\,d\lambda\,.$$

2.14 The moment generating function ϕ of a real-valued random variable X is defined by

$$\phi(\theta) = \mathbb{E}\big(e^{\theta X}\big)\,, \quad \theta \in \mathbb{R}.$$

(a) Show that $I = \{\theta : \phi(\theta) < \infty\}$ is an interval and find examples for $I = \mathbb{R}$, $\{0\}$ and $(-\infty, 1)$.

Assume for simplicity that $X \geq 0$ from now on.

(b) Show that if I contains a neighbourhood of 0 then X has finite moments of all orders given by $\quad \mathbb{E}(X^n) = \left(\frac{d}{d\theta}\right)^n\Big|_{\theta=0}\phi(\theta)\,.$

(c) Find a necessary and sufficient condition on the sequence of moments $m_n = \mathbb{E}(X^n)$ for I to contain a neighbourhood of 0.