

Probability and Measure

Example sheet 1

1.1 (a) Let $\{\mathcal{A}_i : i \in I\}$ be a collection of σ -algebras. Show that $\mathcal{A} = \bigcap_{i \in I} \mathcal{A}_i$ is also a σ -algebra. (You should not assume that I is countable.)

(b) Give an example of σ -algebras $\mathcal{A}_1, \mathcal{A}_2$, such that $\mathcal{A}_1 \cup \mathcal{A}_2$ is not a σ -algebra.

1.2 Let Ω be a set.

(a) Let $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ be the set of all finite sets and their complements (called *cofinite* sets). Show that \mathcal{F} is an algebra.

(b) Let $\mathcal{G} \subseteq \mathcal{P}(\Omega)$ be the set of all countable sets and their complements. Show that \mathcal{G} is a σ -algebra.

1.3 Show that the following families of subsets of \mathbb{R} generate the same σ -algebra \mathcal{B} :

(i) $\{(a, b) : a < b\}$, (ii) $\{(a, b] : a < b\}$, (iii) $\{(-\infty, b] : b \in \mathbb{R}\}$.

1.4 A σ -algebra is called *separable* if it can be generated by a countable family of sets. Show that the Borel σ -algebra \mathcal{B} of \mathbb{R} is separable.

1.5 Let \mathcal{A} be a ring on Ω and $\mu : \mathcal{A} \rightarrow [0, \infty]$ an additive set function. Show that:

(a) If μ is continuous from below at all $A \in \mathcal{A}$ it is also countably additive.

(b) If μ is countably additive it is also countably **sub**additive.

1.6 Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of sets in \mathcal{A} , and define

$$\liminf A_n = \bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} A_m, \quad \limsup A_n = \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} A_m.$$

(a) Show that $\mu(\liminf A_n) \leq \liminf_{n \rightarrow \infty} \mu(A_n)$.

(b) Show that $\mu(\limsup A_n) \geq \limsup_{n \rightarrow \infty} \mu(A_n)$ if $\mu(\Omega) < \infty$.

Give an example with $\mu(\Omega) = \infty$ when this inequality fails.

1.7 Completion

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. A subset N of Ω is called *null* if $N \subseteq B$ for some $B \in \mathcal{A}$ with $\mu(B) = 0$. Write \mathcal{N} for the set of all null sets.

(a) Prove that the family of subsets $\mathcal{C} = \{A \cup N : A \in \mathcal{A}, N \in \mathcal{N}\}$ is a σ -algebra.

(b) Show that the measure μ may be extended to a measure μ' on \mathcal{C} with $\mu'(A \cup N) = \mu(A)$.

The σ -algebra \mathcal{C} is called the *completion* of \mathcal{A} with respect to μ .

1.8 Show that a π -system which is also a d -system is a σ -algebra.

1.9 Let $B \subseteq \mathbb{R}$ be a Borel subset with $\lambda(B) < \infty$, where λ is the Lebesgue measure. Show that, for every $\epsilon > 0$, there exists a finite union of disjoint intervals $A = (a_1, b_1] \cup \dots \cup (a_n, b_n]$ such that $\lambda(A \Delta B) < \epsilon$, where $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

1.10 Cantor set

Let $C_0 = [0, 1]$, and let C_1, C_2, \dots be constructed iteratively by deletion of middle-thirds. Thus,

$$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1], \quad C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1],$$

and so on. The set $C = \lim_{n \rightarrow \infty} C_n = \bigcap_{n \in \mathbb{N}} C_n$ is called the *Cantor set*.

Let F_n be the distribution function of the uniform probability measure concentrated on C_n .

- (a) Show that C is uncountable and has Lebesgue measure 0.
- (b) Show that the limit $F(x) = \lim_{n \rightarrow \infty} F_n(x)$ exists for all $x \in [0, 1]$.
- (c) Show that F is continuous on $[0, 1]$ with $F(0) = 0, F(1) = 1$.
- (d) Show that F is differentiable except on a set of measure 0, and that $F'(x) = 0$ wherever F is differentiable.

1.11 Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of events in the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, i.e. $A_n \in \mathcal{A}$ for all n . Show that the $A_n, n \in \mathbb{N}$, are independent if and only if the σ -algebras which they generate, $\mathcal{A}_n = \{\emptyset, A_n, A_n^c, \Omega\}$, are independent.

1.12 Let μ_F be the Lebesgue-Stieltjes measure on \mathbb{R} associated with the distribution function F .

- (a) Show that F is continuous at x if and only if $\mu_F(\{x\}) = 0$.
- (b) What is the corresponding condition on the so-called ‘joint distribution function’ F when working in \mathbb{R}^d with $d \geq 2$?

1.13 Let $(F_n)_{n \in \mathbb{N}}$, be a sequence of distribution functions on \mathbb{R} such that the limit $F(x) = \lim_{n \rightarrow \infty} F_n(x)$ exists for all $x \in \mathbb{R}$. Show that F need not be a distribution function.

1.14 Riemann zeta function

The *Riemann zeta function* is given by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad s > 1$.

Let $s > 1$ and $\mathbb{P}_f : \mathcal{P}(\mathbb{N}) \rightarrow [0, 1]$ be the probability measure with mass function $f(n) = n^{-s}/\zeta(s)$. For $p \in \{1, 2, \dots\}$ let $A_p = \{n \in \mathbb{N} : p \mid n\}$ (p divides n).

- (a) Show that the events $\{A_p : p \text{ prime}\}$ are independent. Deduce Euler’s formula

$$\frac{1}{\zeta(s)} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right).$$

- (b) Show that $\mathbb{P}_f(\{n \in \mathbb{N} : n \text{ is square-free}\}) = \frac{1}{\zeta(2s)}$.