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www.statslab.cam.ac.uk/∼stefan/teaching/probmeas.html Part IID, M 2005

Probability and Measure

Hand-out 3

Ergodic Theorems

Let $(E, \mathcal{E}, \mu)$ be a σ-finite measure space with a meas.-preserving transformation $\theta : E \to E$.
Let $f : E \to \mathbb{R}$ be integrable and define $S_n : E \to \mathbb{R}$ by $S_0 = 0$ and

$$S_n = S_n(f) = f + f \circ \theta + \ldots + f \circ \theta^{n-1} \quad \text{for } n \geq 1 .$$

Maximal ergodic lemma

Let $S^* = \sup_{n\in\mathbb{N}} S_n : E \to \mathbb{R}$. *Then* $\int_{S^*>0} f\, d\mu \geq 0$.

Proof. Set $S_n^* = \max_{0\leq m\leq n} S_m$ and $A_n = \{S_n^* > 0\}$. Then, for $m = 1, \ldots, n$,

$$S_m = f + S_{m-1} \circ \theta \leq f + S_n^* \circ \theta .$$

On A_n we have $S_n^* = \max_{1\leq m\leq n} S_m$, so $S_n^* \leq f + S_n^* \circ \theta$.
On A_n^c we have $S_n^* = 0 \leq S_n^* \circ \theta$. So, integrating and adding, we obtain

$$\int_E S_n^*\, d\mu \leq \int_{A_n} f\, d\mu + \int_E S_n^* \circ \theta\, d\mu .$$

But S_n^* is integrable and θ is measure-preserving, so

$$\int_E S_n^* \circ \theta\, d\mu = \int_E S_n^*\, d\mu < \infty \quad \text{which implies} \quad \int_{A_n} f\, d\mu \geq 0 .$$

As $n \to \infty$, $A_n \nearrow \{S^* > 0\}$ so, by monotone convergence, $\int_{\{S^*>0\}} f\, d\mu \geq 0$. □

Birkhoff's almost everywhere ergodic theorem

There exists $\bar{f} : E \to \mathbb{R}$ *invariant, with* $\int_E |\bar{f}|\, d\mu \leq \int_E |f|\, d\mu$ *and* $\frac{S_n}{n} \to \bar{f}$ *a.e. as* $n \to \infty$.

Proof. The functions $\liminf_n (S_n/n)$ and $\limsup_n (S_n/n)$ are invariant, since

$$\Big(\liminf \frac{S_n}{n}\Big) \circ \theta = \liminf \Big(\frac{S_n \circ \theta}{n}\Big) = \liminf \Big(\frac{S_{n+1} - f}{n}\Big) = \liminf \Big(\frac{S_{n+1}}{n+1}\Big) .$$

Therefore, for $a < b$,

$$D = D(a, b) = \big\{ \liminf_n (S_n/n) < a < b < \limsup_n (S_n/n) \big\} .$$

is an invariant event.

We shall show that $\mu(D) = 0$. First, by invariance, we can restrict everything to D and thereby reduce to the case $D = E$. Note that either $b > 0$ or $a < 0$. We can interchange the two cases by replacing f by $-f$. Let us assume then that $b > 0$.
Let $B \in \mathcal{E}$ with $\mu(B) < \infty$, then $g = f - b\,1\!\!1_B$ is integrable and, for each $x \in D$, for some n,

$$S_n(g)(x) \geq S_n(f)(x) - n\,b > 0\ .$$

Hence $S^*(g) > 0$ everywhere and, by the maximal ergodic lemma,

$$0 \leq \int_D (f - b\,1\!\!1_B)\,d\mu = \int_D f\,d\mu - b\,\mu(B)\ .$$

Since μ is σ-finite, we can let $B \nearrow D$ to obtain $\quad b\,\mu(D) \leq \int_D f\,d\mu$.
In particular, we see that $\mu(D) < \infty$. A similar argument applied to $-f$ and $-a$, this time with $B = D$, shows that $\quad (-a)\mu(D) \leq \int_D (-f)\,d\mu$. $\quad$ Hence $\quad b\,\mu(D) \leq \int_D f\,d\mu \leq a\,\mu(D)$.
Since $a < b$ and the integral is finite, this forces $\mu(D) = 0$.
Back to general E. Set

$$\Delta = \left\{ \liminf_n (S_n/n) < \limsup_n (S_n/n) \right\}\ ,$$

then Δ is invariant. Also, $\Delta = \bigcup_{a,b\in\mathbb{Q},a<b} D(a,b)$, so $\mu(\Delta) = 0$. On the complement of Δ, S_n/n converges in $[-\infty, \infty]$, so we can define an invariant function $\bar{f}$ by

$$\bar{f} = \begin{cases} \lim_n (S_n/n) & \text{on } \Delta^c \\ 0 & \text{on } \Delta \end{cases}\ .$$

Finally, we have $\mu\big(|f \circ \theta^n|\big) = \mu\big(|f|\big)$, so $\mu\big(|S_n|\big) \leq n\,\mu\big(|f|\big)$ for all n. Hence, by Fatou's lemma,

$$\mu\big(|\bar{f}|\big) = \mu\big(\liminf_n |S_n/n|\big) \leq \liminf_n \mu\big(|S_n/n|\big) \leq \mu\big(|f|\big)\ . \qquad \square$$

Von Neumann's L^p ergodic theorem
Assume that $\mu(E) < \infty$ and $p \in [1, \infty)$. Then, for $f \in L^p$, $S_n/n \to \bar{f}$ (from Birkhoff) in L^p.

Proof. Since θ is measure-preserving we have

$$\|f \circ \theta^n\|_p = \left(\int_E |f|^p \circ \theta^n\,d\mu\right)^{1/p} = \|f\|_p\ .$$

So, by Minkowski's inequality, $\qquad \|S_n(f)/n\|_p \leq \|f\|_p$.
Given $\epsilon > 0$, choose $K < \infty$ so that $\|f - g\|_p < \epsilon/3$, where $g = (-K) \vee f \wedge K$.
By Birkhoff's theorem, $S_n(g)/n \to \bar{g}$ *a.e.* . We have $|S_n(g)/n| \leq K$ for all n so, by bounded convergence, there exists N such that, for $n \geq N$,

$$\|S_n(g)/n - \bar{g}\|_p < \epsilon/3\ .$$

By Fatou's lemma,

$$\|\bar{f} - \bar{g}\|_p^p = \int_E \liminf_n \left|\frac{S_n(f-g)}{n}\right|^p d\mu \leq \liminf_n \int_E \left|\frac{S_n(f-g)}{n}\right|^p d\mu \leq \|f - g\|_p^p\ .$$

Hence, for $n \geq N$,

$$\left\|\frac{S_n(f)}{n} - \bar{f}\right\|_p \leq \left\|\frac{S_n(f-g)}{n}\right\|_p + \left\|\frac{S_n(g)}{n} - \bar{g}\right\|_p + \|\bar{g} - \bar{f}\|_p < 3\epsilon/3 = \epsilon\ . \qquad \square$$