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www.statslab.cam.ac.uk/∼stefan/teaching/probmeas.html

Probability and Measure

Hand-out 1

Proof of Dynkin's π-system lemma.
Denote by $\mathcal{D}$ the intersection of all d-systems containing $\mathcal{A}$. Then $\mathcal{D}$ is itself a d-system. We shall show that $\mathcal{D}$ is also a π-system and hence a σ-algebra, thus proving the lemma. Consider

$$\mathcal{D}' = \big\{ B \in \mathcal{D} \,:\, B \cap A \in \mathcal{D} \text{ for all } A \in \mathcal{A} \big\} \ .$$

Then $\mathcal{A} \subseteq \mathcal{D}'$ because $\mathcal{A}$ is a π-system. Let us check that $\mathcal{D}'$ is a d-system: clearly $\Omega \in \mathcal{D}'$; next, suppose $B_1, B_2 \in \mathcal{D}'$ with $B_1 \subseteq B_2$, then for $A \in \mathcal{A}$ we have $(B_2 \setminus B_1) \cap A = (B_2 \cap A) \setminus (B_1 \cap A) \in \mathcal{D}$, because $\mathcal{D}$ is a d-system, so $B_2 \setminus B_1 \in \mathcal{D}'$; finally, if $B_n \in \mathcal{D}', n \in \mathbb{N}$, and $B_n \nearrow B$, then for $A \in \mathcal{A}$ we have $B_n \cap A \nearrow B \cap A$ so $B \cap A \in \mathcal{D}$ and $B \in \mathcal{D}'$. Hence $\mathcal{D} = \mathcal{D}'$.
Now consider

$$\mathcal{D}'' = \big\{ B \in \mathcal{D} \,:\, B \cap A \in \mathcal{D} \text{ for all } A \in \mathcal{D} \big\} \ .$$

Then $\mathcal{A} \subseteq \mathcal{D}''$ because $\mathcal{D} = \mathcal{D}'$. We can check that $\mathcal{D}''$ is a d-system, just as we did for $\mathcal{D}'$. Hence $\mathcal{D}'' = \mathcal{D}$ which shows that $\mathcal{D}$ is a π-system. □

Proof of Carathéodory's extension theorem.
For any $B \subseteq \Omega$, define the *outer measure* $\quad \mu^*(B) = \inf \sum_n \mu(A_n)$,

where the infimum is taken over all sequences $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ such that $B \subseteq \bigcup_n A_n$ and is taken to be ∞ if there is no such sequence. Note that μ^* is increasing and $\mu^*(\emptyset) = 0$. Let us say that $A \subseteq \Omega$ is *μ^*-measurable* if, for all $B \subseteq \Omega$,

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c) \ .$$

Write $\mathcal{M}$ for the set of all μ^*-measurable sets. *We shall show that $\mathcal{M}$ is a σ-algebra containing $\mathcal{A}$ and that μ^* is a measure on $\mathcal{M}$, extending μ.* This will prove the theorem.

Step I. We show that μ^* is countably subadditive.
Suppose that $B \subseteq \bigcup_n B_n$. If $\mu^*(B_n) < \infty$ for all n, then, given $\epsilon > 0$, there exist sequences $(A_{nm})_{m\in\mathbb{N}}$ in $\mathcal{A}$, with

$$B_n \subseteq \bigcup_m A_{nm}, \quad \mu^*(B_n) + \epsilon/2^n \geq \sum_m \mu(A_{nm}) \ .$$

Then $\quad B \subseteq \bigcup_n \bigcup_m A_{nm} \quad$ and thus $\quad \mu^*(B) \leq \sum_n \sum_m \mu(A_{nm}) \leq \sum_n \mu^*(B_n) + \epsilon$.

Hence, in any case $\quad \mu^*(B) \leq \sum_n \mu^*(B_n)$.

Step II. We show that μ^* extends μ.
Since $\mathcal{A}$ is a ring and μ is countably additive, μ is countably subadditive. Hence, for $A \in \mathcal{A}$ and any sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ with $A \subseteq \bigcup_n A_n$, we have $\quad \mu(A) \leq \sum_n \mu(A_n)$.

On taking the infimum over all such sequences, we see that $\mu(A) \leq \mu^*(A)$. On the other hand, it is obvious that $\mu^*(A) \leq \mu(A)$ for $A \in \mathcal{A}$.

Step III. We show that $\mathcal{M}$ contains $\mathcal{A}$.
Let $A \in \mathcal{A}$ and $B \subseteq \Omega$. We have to show that

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c) \ .$$

By subadditivity of μ^*, it is enough to show that

$$\mu^*(B) \geq \mu^*(B \cap A) + \mu^*(B \cap A^c) \ .$$

If $\mu^*(B) = \infty$, this is trivial, so let us assume that $\mu^*(B) < \infty$. Then, given $\epsilon > 0$, we can find a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ such that

$$B \subseteq \bigcup_n A_n \ , \quad \mu^*(B) + \epsilon \geq \sum_n \mu(A_n) \ .$$

Then $\quad B \cap A \subseteq \bigcup_n (A_n \cap A) \ , \quad B \cap A^c \subseteq \bigcup_n (A_n \cap A^c) \ ,$ so that

$$\mu^*(B \cap A) + \mu^*(B \cap A^c) \leq \sum_n \mu(A_n \cap A) + \sum_n \mu(A_n \cap A^c) = \sum_n \mu(A_n) \leq \mu^*(B) + \epsilon \ .$$

Since $\epsilon > 0$ was arbitrary, we are done.

Step IV. We show that $\mathcal{M}$ is an algebra.
Clearly $\Omega \in \mathcal{M}$ and $A^c \in \mathcal{A}$ whenever $A \in \mathcal{A}$. Suppose that $A_1, A_2 \in \mathcal{M}$ and $B \subseteq \Omega$. Then

$$\begin{aligned}
\mu^*(B) &= \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c) \\
&= \mu^*(B \cap A_1 \cap A_2) + \mu^*(B \cap A_1 \cap A_2^c) + \mu^*(B \cap A_1^c) \\
&= \mu^*(B \cap A_1 \cap A_2) + \mu^*(B \cap (A_1 \cap A_2)^c \cap A_1) + \mu^*(B \cap (A_1 \cap A_2)^c \cap A_1^c) \\
&= \mu^*(B \cap (A_1 \cap A_2)) + \mu^*(B \cap (A_1 \cap A_2)^c) \ .
\end{aligned}$$

Hence $A_1 \cap A_2 \in \mathcal{M}$.

Step V. We show that $\mathcal{M}$ is a σ-algebra and that μ^* is a measure on $\mathcal{M}$.
We already know that $\mathcal{M}$ is an algebra, so it suffices to show that, for any sequence of disjoint sets $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$, for $A = \bigcup_n A_n$ we have

$$A \in \mathcal{M} \ , \quad \mu^*(A) = \sum_n \mu^*(A_n).$$

So, take any $B \subseteq \Omega$, then

$$\begin{aligned}
\mu^*(B) &= \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c) = \mu^*(B \cap A_1) + \mu^*(B \cap A_2) + \mu^*(B \cap A_1^c \cap A_2^c) \\
&= \ldots = \sum_{i=1}^n \mu^*(B \cap A_i) + \mu^*(B \cap A_1^c \cap \ldots \cap A_n^c) \ .
\end{aligned}$$

Note that $\mu^*(B \cap A_1^c \cap \ldots \cap A_n^c) \geq \mu^*(B \cap A^c)$ for all n. Hence, on letting $n \to \infty$ and using countable subadditivity, we get

$$\mu^*(B) \geq \sum_{n=1}^\infty \mu^*(B \cap A_n) + \mu^*(B \cap A^c) \geq \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

The reverse inequality holds by subadditivity, so we have equality. Hence $A \in \mathcal{M}$ and, setting $B = A$, we get $\quad \mu^*(A) = \sum_{n=1}^\infty \mu^*(A_n) \ .$ □