# On the Cahn-Hilliard Equation

CHARLES M. ELLIOTT & ZHENG SONGMU

*Communicated by* J. B. MCLEOD

## § 1. Introduction

The phenomenological Cahn-Hilliard equation

$$\frac{\partial u}{\partial t} + \gamma \frac{\partial^4 u}{\partial x^4} = \frac{\partial^2 \varphi(u)}{\partial x^2}, \quad 0 < x < L, \quad 0 < t \tag{1–1a}$$

$$\varphi(u) = \gamma_2 u^3 + \gamma_1 u^2 - u \tag{1–1b}$$

where $\gamma$, $\gamma_1$ and $\gamma_2$ are constants with $\gamma > 0$, arises in the study of phase separation in cooling binary solutions such as alloys, glasses and polymer mixtures; see CAHN & HILLIARD [1958], NOVICK-COHEN & SEGEL [1984], NOVICK-COHEN [1985] and the references cited therein. Here $u(x, t)$ is a perturbation of the concentration of one of the phases and (1–1a) is the equation of conservation of mass with the mass flux $J$ being

$$J = -\frac{\partial}{\partial x}\left[\varphi(u) - \gamma \frac{\partial^2 u}{\partial x^2}\right]. \tag{1–2}$$

Clearly critical points of the Landau-Ginzburg free energy form,

$$\int_0^L \left\{H(u) + \tfrac{1}{2}\gamma\left(\frac{\partial u}{\partial x}\right)^2\right\} dx, \tag{1–3a}$$

$$H(u) = \int_0^u \varphi(s)\, ds, \tag{1–3b}$$

with appropriate side conditions are steady state solutions of (1–1). See CARR, GURTIN & SLEMROD [1984] for the study of (1–3) for small $\gamma$ and subject to the constraint of prescribed mass,

$$\frac{1}{L}\int_0^L u(x)\, dx = M. \tag{1–4}$$

Equation (1–1) is supplemented by the zero mass flux boundary condition

$$-\frac{\partial\varphi(u)}{\partial x}+\gamma\frac{\partial^3 u}{\partial x^3}\bigg|_{x=0,L}=0, \tag{1–5a}$$

the natural boundary condition for (1–3),

$$\frac{\partial u}{\partial x}\bigg|_{x=0,L}=0 \tag{1–5b}$$

and the initial condition

$$u(x,0)=u_0(x) \qquad 0<x<L. \tag{1–5c}$$

It follows from (1–5b) and (1–1b) that (1–5a) can be replaced by

$$\frac{\partial^3 u}{\partial x^3}\bigg|_{x=0,L}=0. \tag{1–5d}$$

A solution of (1–1) and (1–5) satisfies

$$\frac{d}{dt}\int_0^L u(x,t)\,dx=\int_0^L\frac{\partial u}{\partial t}(x,t)\,dx=\int_0^L-\frac{\partial J}{\partial x}\,dx=0$$

and hence the total mass remains constant,

$$\frac{1}{L}\int_0^L u(x,t)\,dx=\frac{1}{L}\int_0^L u_0(x)\,dx=M, \qquad t>0. \tag{1–6}$$

Equation (1–1) has been considered in other contexts in order to generate spatial pattern formation. COHEN & MURRAY [1981] derive it, in an ecological context, as a generalization of Fickian diffusion. HAZEWINKEL, KAASHOEK & LEYNSE [1985] obtain the equation as a limit of THOM's river basin model.

In this paper we consider the global existence or blow up in a finite time of the solution to the initial boundary value problem (1–1) and (1–5) and its related finite element Galerkin approximation. We have found that the sign of $\gamma_2$ in (1–1b) is crucial. If $\gamma_2>0$, then there is a unique global solution for any initial data $u_0\in H^2$ and satisfying (1–5b). If $\gamma_2<0$, then the solution must blow up in a finite time for large initial data. On the other hand, if $\gamma>\frac{L^2}{\pi^2}$ and the initial data is small, no matter what the sign of $\gamma_2$ is, there is a unique global solution which decays to the constant $M$ as $t\to\infty$. We also extend these results to the multidimensional problem.

$$\frac{\partial u}{\partial t}+\gamma\,\Delta^2 u=\Delta\varphi(u) \qquad x\in\Omega, \quad t>0 \tag{1–7a}$$

$$\frac{\partial u}{\partial \nu}=0, \quad \frac{\partial}{\partial\nu}(\gamma\,\Delta u-\varphi(u))=0 \qquad x\in\Gamma, \quad t>0 \tag{1–7b}$$

$$u(x,0)=u_0(x), \qquad x\in\Omega \tag{1–7c}$$

where $\Gamma$ is the smooth boundary of a bounded domain $\Omega$ in $\mathbb{R}^n$ ($n \leqq 3$) and $\frac{\partial}{\partial \nu}$ is the exterior normal derivative to $\Gamma$. The global existence theorems are proved in section 2 and finite time blow up is obtained in section 3.

In the remaining section we study a finite element Galerkin approximation to the initial boundary value problem and obtain existence results and optimal order error bounds.

Throughout the paper we use $D$ to denote $\frac{\partial}{\partial x}$ and $Q_T$ to denote $\Omega \times (0, T)$. The norms of $L^\infty(\Omega)$, $L^2(\Omega)$ and $H^s(\Omega)$ are denoted by $\|\cdot\|_\infty$, $\|\cdot\|$ and $\|\cdot\|_s$. The semi-norm $\|D^s v\|$ is denoted by $|v|_s$.

We note the Friedrichs inequality

$$\|v\| \leqq \begin{cases} L/\pi\, |v|_1, & n = 1 \\ C(\Omega)\, |v|, & n \geqq 2 \end{cases} \quad \forall v \in H_0^1(\Omega) \tag{1-8}$$

the Poincaré inequality

$$\|v\|^2 \leqq \begin{cases} \frac{L^2}{2} |v|_1^2 + 1/L \left(\int_0^L v\, dx\right)^2, & n = 1 \\ C(\Omega)\left\{|v|_1^2 + \left(\int_\Omega v(x)\, dx\right)^2\right\}, & n \geqq 2, \end{cases} \quad \forall\, v \in H^1(\Omega), \tag{1-9}$$

and the Nirenberg inequality (see ADAMS [1975])

$$\|D^j v\|_{L^p} \leqq C_1 \|D^m v\|_{L^r}^a \|v\|_{L^q}^{1-a} + C_2 \|v\|_{L^q}, \tag{1-10a}$$

$$\frac{j}{m} \leqq a \leqq 1, \quad \frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}. \tag{1-10b}$$

Finally, we use the notation $H_E^2(\Omega) = \left\{v \in H^2(\Omega): \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma\right\}$ and note the inequality

$$|v|_1^2 \leqq \|v\|\, \|\Delta v\| \qquad \forall\, v \in H_E^2(\Omega) \tag{1-11}$$

which follows from the equality

$$0 = \int_\Omega \nabla(u \nabla u)\, dx = \int_\Omega \{|\nabla u|^2 + u \Delta u\}\, dx.$$

## § 2. Global Existence

In this section we are going to prove the global existence of solutions to the following initial-boundary value problem:

$$\frac{\partial u}{\partial t} + \gamma D^4 u = D^2 \varphi(x) \quad 0 < x < L, \quad 0 < t < T, \quad I = (0, L) \tag{2-1a}$$

$$Du(0, t) = Du(L, t) = 0, \quad D^3 u(0, t) = D^3 u(L, t) = 0, \quad t > 0 \tag{2-1b}$$

$$u(x, 0) = u_0(x), \quad 0 < x < L \tag{2-1c}$$

where

$$\varphi(u) = -u + \gamma_1 u^2 + \gamma_2 u^3 \tag{2–2}$$

with $\gamma$, $\gamma_1$ and $\gamma_2$ being constants and $\gamma$ being positive. We can easily obtain local in time existence and uniqueness results. It is sufficient to apply the standard Picard iteration scheme. Therefore in order to obtain existence on $[0, T]$ for any $T > 0$ we need *a priori* estimates on $u$.

**Theorem 2.1.** *If* $\gamma_2 > 0$, *then for any initial data* $u_0 \in H_E^2(I)$ *and* $T > 0$ *there exists a unique global solution* $H^{4,1}(Q_T)$. *Moreover, if* $u_0 \in H^6(I) \cap H_E^2(I)$ *and* $D^2 u_0 \in H_E^2(I)$, *then the solution is a classical one.*

**Proof.** Multiplying equation (2-1a) by $u$ and integrating with respect to $x$ we obtain

$$\tfrac{1}{2}\frac{d}{dt}\|u\|^2 + \gamma\|D^2u\|^2 + \int_0^L \varphi'(u)\,(Du)^2\,dx = 0. \tag{2–3}$$

Since $\gamma_2 > 0$, a simple calculation shows that

$$\varphi'(u) = 3\gamma_2 u^2 + 2\gamma_1 u - 1 \geqq -c_0 = -\frac{\gamma_1^2}{3\gamma_2} - 1, \qquad c_0 > 0. \tag{2–4}$$

Thus it follows from (2–3) that

$$\begin{aligned}\tfrac{1}{2}\frac{d}{dt}\|u\|^2 + \gamma\|D^2u\|^2 &\leqq c_0\|Du\|^2\\ &\leqq c_0\|D^2u\|\,\|u\|\\ &\leqq \frac{\gamma}{2}\|D^2u\|^2 + \frac{c_0^2}{\gamma}\|u\|^2,\end{aligned} \tag{2–5}$$

where we have used the inequality (1–11). By the Gronwall inequality, (2–5) implies that

$$\|u(t)\|^2 \leqq \|u_0\|^2\, e^{c_0^2 T/\gamma}, \qquad 0 \leqq t \leqq T \tag{2–6a}$$

$$\int_0^t \|D^2u\|^2\,d\tau \leqq \frac{\|u_0\|^2}{\gamma}\, e^{c_0^2 T/\gamma}, \quad 0 \leqq t \leqq T. \tag{2–6b}$$

In the following we use $C_T$ generically to denote constants depending on $T$ but independent of the solution $u$.

Defining

$$H(u) = \int_0^u \varphi(s)\,ds = \frac{\gamma_2}{4}u^4 + \frac{\gamma_1}{3}u^3 - \tfrac{1}{2}u^2 \tag{2–7a}$$

and

$$F(t) = \int_0^L \left(H(u) + \frac{\gamma}{2}(Du)^2\right) dx, \tag{2–7b}$$

we have

$$\frac{dF}{dt} = \int_0^L \left(\varphi(u)\frac{\partial u}{\partial t} + \gamma\, Du\, D\frac{\partial u}{\partial t}\right) dx. \tag{2–8}$$

Integrations by parts and equations (2–1 a, b) yield

$$\begin{aligned}\frac{dF}{dt} &= \int_0^L [\varphi(u)(-\gamma\, D^4u + D^2\varphi) - \gamma\, D^2u(-\gamma\, D^4u + D^2\varphi)]\, dx\\ &= -\int_0^L [\gamma^2(D^3u)^2 - 2\gamma\, D^3u\, D\varphi + (D\varphi)^2]\, dx\\ &= -\int_0^L [\gamma\, D^3u - D\varphi]^2\, dx \leqq 0,\end{aligned} \tag{2–9}$$

and

$$F(t) \leqq F(0) = \int_0^L \left(H(u_0) + \frac{\gamma}{2}(Du_0)^2\right) dx. \tag{2–10}$$

By Young's inequality

$$u^2 \leqq \varepsilon u^4 + C_{1\varepsilon}, |u^3| \leqq \varepsilon u^4 + C_{2\varepsilon} \tag{2–11}$$

we have from (2–7b), (2–10) and (2–6a) that

$$\frac{\gamma}{2}\|Du\|^2 + \frac{\gamma_2}{2}\left[\int_0^L u^4\, dx + \int_0^L u^2\, dx\right] \leqq C_3 + F(0) = C. \tag{2–12}$$

By Sobolev's imbedding theorem it follows from (2–6a) and (2–12) that

$$\|u(t)\|_\infty \leqq C', \quad \forall\, t \in [0, T]. \tag{2–13}$$

Next we multiply equation (2–1 a) by $D^4u$ and integrate with respect to $x$, obtaining

$$\tfrac{1}{2}\frac{d}{dt}\|D^2u\|^2 + \gamma\, \|D^4u\|^2 = \int_0^L D^2\varphi(u)\, D^4u\, dx. \tag{2–14}$$

Note that

$$\begin{aligned}D^2\varphi(u) &= \varphi'(u)\, D^2u + \varphi''(Du)^2\\ &= (3\gamma_2u^2 + 2\gamma_1 u - 1)\, D^2u + (6\gamma_2 u + 2\gamma_1)(Du)^2.\end{aligned} \tag{2–15}$$

By the Nirenberg inequality (1–10),

$$\|Du\|_\infty \leqq C(\|D^4u\|^{3/8}\, \|u\|^{5/8} + \|u\|), \tag{2–16}$$

we obtain, using (2–12) and (2–13), the inequality

$$\left|\int_0^L \varphi''(u)\,(Du)^2\, D^4u\, dx\right| \leqq C_T \|Du\|_\infty \|Du\| \, \|D^4u\|$$

$$\leqq C_T(\|D^4u\|^{3/8} + 1)\, \|D^4u\|$$

$$\leqq \frac{\gamma}{4} \|D^4u\|^2 + C_T. \tag{2–17}$$

It follows from (2–14), (2–15), (2–17) and (2–13) that

$$\tfrac{1}{2}\frac{d}{dt}\|D^2u\|^2 + \gamma \|D^4u\|^2 \leqq \left|\int_0^L \varphi'(u)\, D^2u\, D^4u\, dx\right| + \left|\int_0^L \varphi''(u)\,(Du)^2\, D^4u\, dx\right|$$

$$\leqq \frac{\gamma}{2}\|D^4u\|^2 + C_T\|D^2u\|^2 \tag{2–18}$$

and by Gronwall's inequality,

$$\|D^2u(t)\|^2 \leqq C_T, \qquad \forall\, t \in [0, T] \tag{2–19a}$$

$$\int_0^t \|D^4u\|^2\, d\tau \leqq C_T, \qquad \forall\, t \in [0, T]. \tag{2–19b}$$

The *a priori* estimates (2–6), (2–12), (2–13) and (2–19) complete the proof of global existence of a $u \in H^{4,1}(Q_T)$.

Further regularity of the solution is obtained by the use of a bootstrap argument. Since $u \in H^{4,1}(Q_T)$ we have

$$Du \in L^\infty(Q_T),\ D^2u \in L^2(0, T; L^\infty(I)). \tag{2–20}$$

from which it follows, by a direct calculation, that

$$f(x, t) \equiv D^2\varphi(u(x, t)),\ Df \in L^2(Q_T),\ D^2f \in L^2(Q_T). \tag{2–21}$$

It is well known (LIONS & MAGENES [1972]) that if $f \in L^2(0, T; L^2(I))$ and $v_0 \in H_E^2(I)$ then the initial boundary value problem

$$\frac{\partial v}{\partial t} + \gamma\, D^4v = f, \tag{2–22a}$$

$$Dv|_{x=0,L} = D^3v|_{x=0,L} = 0,\ v|_{t=0} = v_0 \tag{2–22b}$$

has a unique solition $v \in H^{4,1}(Q_T)$. Now it is easy to see that taking

$$f(x, t) \equiv D^3\varphi(u(x, t)), \qquad v_0 = Du_0 \qquad \text{yields} \qquad v = Du \in H^{4,1}(Q_T), \tag{2–23a}$$

$$f(x, t) \equiv D^4\varphi(u(x, t)), \qquad v_0 = D^2u_0 \qquad \text{yields} \qquad v = D^2u \in H^{4,1}(Q_T). \tag{2–23b}$$

Furthermore, (2–23) implies that $f = \frac{\partial}{\partial t} D^2\varphi \in L^2(Q_T)$ and assuming that $D^5 u_0|_{x=0,L} = 0$ we have that $v_0 = -\gamma\, D^4 u_0 + D^2\varphi(u_0) \in H_E^2(I)$. Hence

$$v = \frac{\partial u}{\partial t} \in H^{4,1}(Q_T) \tag{2–24}$$

and by interpolation theory, (2–23) and (2–24) imply that

$$Du,\, D^4 u \in C(\overline{Q}_T). \tag{2–25}$$

This completes the proof of the existence of a classical solution. □

We turn now to the proof of global existence for $\gamma$ sufficiently large and $\|u_0\|_2$ sufficiently small. Note that integration of (2–1) yields

$$\frac{1}{L}\int_0^L u(x, t)\, dx = \frac{1}{L}\int_0^L u_0(x)\, dx \equiv M. \tag{2–26}$$

If we set

$$v(x, t) = u(x, t) - M, \tag{2–27}$$

so that

$$\int_0^L v(x, t)\, dx = 0, \tag{2–28}$$

the problem (2–1) is converted into

$$\frac{\partial v}{\partial t} + \gamma\, D^4 v = D^2\tilde{\varphi}(v), \tag{2–29a}$$

$$Dv|_{x=0,L} = D^3 v|_{x=0,L} = 0, \tag{2–29b}$$

$$v(x, 0) = u_0(x) - M, \tag{2–29c}$$

where

$$\tilde{\varphi}(v) = \gamma_2 v^3 + (3\gamma_2 M + \gamma_1)\, v^2 + (3\gamma_2 M^2 + 2\gamma_1 M - 1)\, v. \tag{2–30}$$

**Theorem 2.2.** *If $\gamma > L^2/\pi^2$, $u_0 \in H_E^2(I)$ and $\|u_0\|_2$ is sufficiently small, then there exists a unique global solution $u \in H^{4,1}(Q_T)$ to (2–1). Moreover, it holds that*

$$\lim_{t\to\infty} \|u(t) - M\|_\infty = \lim_{t\to\infty} \|Du(t)\|_\infty = \lim_{t\to\infty} \|D^2 u(t)\| = 0. \tag{2–31}$$

**Proof.** It is easy to see that problem (2–1) is equivalent to (2–29). As previously noted we have local in time existence and uniqueness of a solution so that for global existence it is only necessary to obtain *a priori* estimates of $v$. In what follows $C_j$, $j = 1, 2, \ldots$ denote constants which are independent of $v$ and $t$. If we set

$$\gamma_0 = 3\gamma_2 M^2 + 2\gamma_1 M - 1, \quad \tilde{\gamma}_1 = 3\gamma_2 M + \gamma_1, \tag{2–32}$$

equation (2–29a) may be rewritten as

$$\frac{\partial v}{\partial t} + \gamma\, D^4 v - \gamma_0\, D^2 v = f \equiv D^2(\gamma_2 v^3 + \tilde{\gamma}_1 v^2). \tag{2–33}$$

Since $\|u_0\|_2$ is assumed to be sufficiently small, we may assume that

$$|\gamma_0| < \gamma\pi^2/L^2. \tag{2–34}$$

Now, for any fixed $t > 0$, define

$$N(t) = \sup_{0<\tau<t} \|v(\tau)\|_2^2 + \int_0^t \|v(\tau)\|_2^2\, d\tau. \tag{2–35}$$

Our goal is to show that $N(t)$ can be bounded, independently of $t$, by the initial data. This is achieved in the following steps.

*Step 1.* Multiplying (2–33) by $v$ and integrating with respect to $x$, we obtain

$$\tfrac{1}{2}\frac{d}{dt}\|v\|^2 + \gamma\,\|D^2 v\|^2 + \gamma_0\,\|Dv\|^2 = \int_0^L fv\, dx. \tag{2–36}$$

Since $Dv \in H_0^1(I)$, Friedrichs' inequality (1–8) implies that

$$\tfrac{1}{2}\frac{d}{dt}\|v\|^2 + C_1\,\|D^2 v\|^2 \leqq \int_0^L fv\, dx \tag{2–37}$$

where

$$C_1 = \gamma - |\gamma_0|\, L^2/\pi^2 > 0. \tag{2–38}$$

Since $\int_0^L v(x, t)\, dx = 0$, by Poincaré's inequality (1–9) and Friedrichs' inequality (1–8) we have

$$\|v\|^2 \leqq C_2\,\|D^2 v\|^2, \tag{2–39}$$

so that (2–37) yields,

$$\tfrac{1}{2}\frac{d}{dt}\|v\|^2 + C_3\,\|v\|_2^2 \leqq C_4\,\|f\|^2. \tag{2–40}$$

*Step 2.* Multiplying (2–33) by $\partial v/\partial t$ and integrating with respect to $x$, we obtain

$$\left\|\frac{\partial v}{\partial t}\right\|^2 + \gamma\frac{d}{dt}\|D^2 v\|^2 + \gamma_0\frac{d}{dt}\|Dv\|^2 \leqq \|f\|^2. \tag{2–41}$$

Integrating (2–41) with respect to $t$, using Friedrichs' inequality (1–8) and noting (2–38) yields

$$\int_0^t \left\|\frac{\partial v}{\partial t}\right\|^2 d\tau + C_1\,\|D^2 v\|^2 \leqq \gamma\,\|D^2 v_0\|^2 + |\gamma_0|\,\|Dv_0\|^2 + \int_0^t \|f\|^2\, d\tau. \tag{2–42}$$

It follows from (2–40) and (2–42) that

$$N(t) \leqq C_4\left\{\|v_0\|_2^2 + \int_0^t \|f\|^2\, d\tau\right\}. \tag{2–43}$$

Since

$$f \equiv D^2(\gamma_2 v^3 + \tilde{\gamma}_1 v^2) = (3\gamma_2 v^2 + 2\tilde{\gamma}_1 v)\, D^2 v + (6\gamma_2 v + 2\tilde{\gamma}_1)\,(Dv)^2,$$

we have

$$\|f\|^2 \leqq C_5\,\{(\|v\|_\infty^4 + \|v\|_\infty^2)\,\|D^2 v\|^2 + (\|v\|_\infty^2\,\|Dv\|_\infty^2 + \|Dv\|_\infty^2)\,\|Dv\|^2\}. \tag{2–44}$$

Sobolev's inequality for one dimension and Poincaré's inequality (1–9) yield

$$\|v\|_\infty \leqq C_6\,\|Dv\|, \qquad \|Dv\|_\infty \leqq C_7\,\|D^2 v\|$$

and from (2–44) we have that

$$\|f\|^2 \leqq C_8(\|D^2 v\|^4 + \|D^2 v\|^6)$$

and

$$\int_0^t \|f\|^2\, d\tau \leqq C_8 \sup_{\tau\in[0,t]} \|v\|_2^2 \left[1 + \sup_{\tau\in[0,t]} \|v\|_2^2\right] \int_0^t \|v\|_2^2\, d\tau. \tag{2–45}$$

Taking (2–43) and (2–45) together yields

$$N(t) \leqq C_9\,\{\|v_0\|_2^2 + N(t)^2 + N(t)^3\} \qquad \forall\, t > 0. \tag{2–46}$$

By considering the graph of the function $F(N) = C_9\{\|v_0\|_2^2 + N^2 + N^3\} - N$ and following the argument of Klainerman & Ponce [1983] it is clear that if $\|v_0\|_2$ is sufficiently small then there is a constant $C_{10}$ such that

$$N(t) \leqq C_{10}\,\|v_0\|_2^2, \qquad \forall\, t > 0. \tag{2–47}$$

This proves the global existence of a weak solution in $H^{2,1}(Q_T)$. To complete the proof of global existence in $H^{4,1}(Q_T)$ we observe that multiplying (2-33) by $-D^2 v$ and $D^4 v$ yield, after calculations similar to the above, the inequalities

$$\|Dv\|^2 + \int_0^t \|D^3 v\|^2\, d\tau \leqq C_{11}\left\{\|v_0\|_1^2 + \int_0^t \|f\|^2\, d\tau\right\} \tag{2–48a}$$

$$\|D^2 v\|^2 + \int_0^t \|D^4 v\|^2\, d\tau \leqq C_{12}\left\{\|v_0\|_2^2 + \int_0^t \|f\|^2\, d\tau\right\}. \tag{2–48b}$$

Thus *a priori* bounds in $H^{4,1}(Q_T)$ follow from (2–48), (2–46) and (2–45).

In order to prove that $v$ tends to zero as $t \to \infty$ we notice that, since (2–47) holds for all $t$,

$$\|f\|^2 \leqq \varepsilon\,\|D^2 v\|^2 \tag{2–49}$$

where $\varepsilon$ is sufficiently small provided $\|v_0\|_2$ is sufficiently small. It follows from (2-40) that

$$\tfrac{1}{2}\frac{d}{dt}\|v\|^2 + (C_3 - \varepsilon C_4)\,\|v\|_2^2 \leqq 0 \tag{2–50}$$

which implies for $\varepsilon(\|v_0\|_2)$ sufficiently small that $\|v\|$ decays exponentially to zero. Similarly, we obtain $\|v\|_2 \to 0$ as $t \to \infty$ from the differential inequalities corresponding to (2-48). Thus we have also that $\|v\|_\infty$ and $\|Dv\|_\infty$ also tend to zero as $t \to \infty$. □

*Remark 1.* If the initial data is close to a constant $M$ and $|\varphi'(M)| < \gamma\pi^2/L^2$ then we have similar results. In particular consider the Sivashinsky equation modelling a planar solid-liquid interface for a binary alloy (SIVASHINSKY [1983])

$$\frac{\partial u}{\partial t} + D^4 u + \alpha u - D^2(2u - \tfrac{1}{2}u^2) = 0, \qquad \alpha > 0, \tag{2–51}$$

with the same initial boundary values (2–1 b, c). If $\pi^2 > 2L^2$ or $\alpha > 1$ then problem (2–51, 2–1 b, c) has a unique global solution provided the initial data is small.

*Remark 2* (Multidimensions $n \leqq 3$). The corresponding problem for $n = 2, 3$ is

$$\frac{\partial u}{\partial t} + \gamma\,\Delta^2 u = \Delta\varphi(u), \tag{2–52a}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial}{\partial \nu}\Delta u = 0, \text{ on } \Gamma \tag{2–52b}$$

$$u|_{t=0} = u_0, \tag{2–52c}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ ($n = 2, 3$) with a smooth boundary $\Gamma$ and $\nu$ is the unit exterior normal to $\Gamma$. For $u_0 \in H_E^2(\Omega)$ there exists a unique global solution $u \in H^{4,1}(Q_T)$. The proof is the same as that of Theorem 2.1 with minor changes. Since under the translation

$$v = u - M, \qquad M = \int_\Omega u_0(x)\,dx/|\Omega| \tag{2–53}$$

the value of $\gamma_2$ does not change, we may, without loss of generality, assume that

$$\int_\Omega u_0(x)\,dx = 0 = \int_\Omega u(x,t)\,dt. \tag{2–54}$$

Now as before in (2–6) and (2–12) we have

$$\|u(t)\|_1 + \int_0^t |u|_2^2\,d\tau \leqq C_T, \qquad \forall\, t \in (0, T]. \tag{2–55}$$

It can be seen from (2–14) that the crucial term to estimate is $\|\Delta\varphi(u)\|$. By the boundary conditions, (2–54) and the Poincaré-Friedrichs inequalities $\|\Delta^2 u\|$ is equivalent to $\|u\|_4$. By Sobolev's imbedding theorem and (2–55) we have

$$\|u\|_{L^q} \leqq C_T \quad \text{for any} \quad q < \infty \quad (n = 2), \tag{2–56a}$$

$$\|u\|_{L^6} \leqq C_T \quad (n = 3). \tag{2–56b}$$

By the Nirenberg inequality (1–10), we have

$$\|u\|_\infty \leqq C\,\|\Delta^2 u\|^a\,\|u\|_{L^q}^{1-a} \quad \text{where } a = (1 + 3q/2)^{-1} \quad (n = 2), \tag{2–57a}$$

$$\|u\|_\infty \leqq C\,\|\Delta^2 u\|^{\frac{1}{6}}\,\|u\|_{L^6}^{\frac{5}{6}} \quad (n = 3), \tag{2–57b}$$

$$\|\nabla u\|_{L^4} \leqq C\,\|\Delta^2 u\|^{\frac{1}{6}}\,\|\nabla u\|^{\frac{5}{6}} \quad (n = 2), \tag{2–58a}$$

$$\|\nabla u\|_{L^4} \leqq C\,\|\Delta^2 u\|^{\frac{1}{4}}\,\|\nabla u\|^{\frac{3}{4}} \quad (n = 3) \tag{2–58b}$$

and

$$\|\Delta u\| \leqq C\,\|\Delta^2 u\|^{\frac{1}{3}}\,\|\nabla u\|^{\frac{2}{3}} \quad (n = 2), \tag{2–59a}$$

$$\|\Delta u\| \leqq C\,\|\Delta^2 u\|^{\frac{1}{2}}\,\|\nabla u\|^{\frac{1}{2}} \quad (n = 3). \tag{2–59b}$$

From these inequalities we finally arrive at

$$\|u^2\,\Delta u\| \leqq \|u\|_\infty^2\,\|\Delta u\| \leqq C_T\,\|\Delta^2 u\|^{\frac{1}{3}+2a} \quad (n = 2), \tag{2–60a}$$

$$\|u^2\,\Delta u\| \leqq C_T\,\|\Delta^2 u\|^{\frac{5}{6}} \quad (n = 3) \tag{2–60b}$$

and

$$\|u\,|\nabla u|^2\| \leqq \|u\|_\infty\,\|\nabla u\|_{L^4}^2 \leqq C_T\,\|\Delta^2 u\|^{a+\frac{1}{3}} \quad (n = 2), \tag{2–61a}$$

$$\|u\,|\nabla u|^2\| \leqq C_T\,\|\Delta^2 u\|^{\frac{2}{3}} \quad (n = 3). \tag{2–61b}$$

Since

$$\Delta\varphi(u) = \varphi'(u)\,\Delta u + \varphi''(u)\,|\nabla u|^2,$$

applying Young's inequality to the right-hand side of

$$\tfrac{1}{2}\frac{d}{dt}\|\Delta u\|^2 + \gamma\,\|\Delta^2 u\|^2 = \int_u \Delta\varphi(u)\,\Delta^2 u\,dx$$

using (2–61), we obtain

$$\|\Delta u(t)\|^2 + \int_0^t \|\Delta^2 u\|^2\,d\tau \leqq C_T, \qquad \forall\, t \in [0, T]. \tag{2–61}$$

This completes the proof of global existence.

## § 3. Blow up in finite time when $\gamma_2 < 0$

In the previous section we proved that if $\gamma_2 > 0$ then (2–1) and (2–5b) admit unique global solutions. On the other hand numerical experiments in one space dimension (HAZEWINKEL, KAASHOEK & LEYNSE [1985]), indicate that if $\gamma_2 < 0$, then, in general, the solution will blow up in finite time. In this section we give a rigorous proof of that.

**Theorem 3.1.** *If* $\gamma_2 < 0$ *and* $-\int_\Omega \left\{H(u_0) + \frac{\gamma}{2}|\nabla u_0|^2\right\} dx$ *is sufficiently large, then the solution* $u$ *of* (2–51) ($n \leqq 3$) *blows up in finite time: there is a* $T^* > 0$ *such that*

$$\lim_{t \to T^*} \|u(t)\|_2 = +\infty. \tag{3–1}$$

**Proof.** Without loss of generality we consider initial data such that (2–54) holds, i.e. $\int_\Omega u_0(x)\,dx = 0$. As in the proof of Theorem 2.1,

$$2\int_\Omega H(u)\,dx - 2F(0) \leqq -\gamma\,|u|_1^2 \tag{3–2}$$

where

$$F(0) = \int_\Omega \left(H(u_0) + \frac{\gamma}{2}|\nabla u_0|^2\right) dx. \tag{3–3}$$

Let $w(x, t)$ be the unique solution of

$$\begin{aligned} &\Delta w = u, \\ &\frac{\partial w}{\partial \nu} = 0, \quad \text{on } \Gamma, \qquad \int_\Omega w\,dx = 0. \end{aligned} \tag{3–4}$$

It follows that

$$|w|_1^2 \leqq C\,\|u\|^2. \tag{3–5}$$

Now multiplying (2–52a) by $w$ and integrating with respect to $x$, using (3–4), we obtain

$$\begin{aligned} \frac{d}{dt}|w|_1^2 &= -2\int_\Omega \varphi(u)\,u\,dx - 2\gamma\,|u|_1^2 \\ &\geqq 4\int_\Omega H(u)\,dx - 4F(0) - 2\int_\Omega \varphi(u)\,u\,dx \\ &= -\gamma_2 \int_\Omega u^4\,dx + \tfrac{10}{3}\gamma_1 \int_\Omega u^3\,dx - 4F(0) \\ &\geqq -\frac{\gamma_2}{2}\int_\Omega u^4\,dx - 4F(0) - C_1 \\ &\geqq \frac{-\gamma_2}{2|\Omega|}\left(\int_\Omega u^2\,dx\right)^2 - 4F(0) - C_1 \end{aligned}$$

and using (3–5),

$$\tfrac{1}{2}\frac{d}{dt}|w|_1^2 \geqq \frac{-\gamma_2}{2|\Omega|\,C}|w|_1^4 - 4F(0) - C_1 \tag{3–6}$$

where $C_1$ is a constant depending only on $\gamma_1$, $\gamma_2$ and $\Omega$. Thus (3–6) yields, when

$$-F(0) > C_1/4, \tag{3–7}$$

that $|w|_1^2$ must blow up in a finite time $T^*$. Hence by (3–5) we have that (3–1) holds. An inspection of the dependence on $u_0$ of $F(0)$ shows that given any $g \in H_E^2(\Omega)$ choosing $u_0 = kg$ yields (3–7) for $k$ large enough.

## § 4. Finite element Galerkin approximation

Let $S_l^r$ be the piecewise polynomial spline space

$$S_l^r = \{\chi \in C^l(I) : \chi|_{I_i} \in \Pi_{r-1}(I_i), \quad i = 1, 2, 3, \dots N\} \tag{4–1}$$

where $r$ and $l$ are integers, $-1 \leqq l \leqq r-1$, $0 = x_0 < x_1 < x_2 < \dots < x_N = L$, $I_i = (x_{i-1}, x_i)$, $|I_i| \in (\delta h, h)$ for some $\delta > 0$ and $\Pi_{r-1}(I_i)$ denotes the set of all polynomials on $I_i$ of degree less or equal to $r-1$. Let $k \geqq 1$ and $r \geqq 4$ so that $S_k^r \subset H^2(I)$ and let $\mathring{S}_k^r \subset H_E^2(I)$ denote $\{\chi : D\chi(0) = D\chi(L) = 0\} \cap S_k^r$. The following approximation property is assumed for all $v \in H_E^2(I) \cap W_p^s(I)$ with $2 \leqq s \leqq r$,

$$\inf_{\chi \in \mathring{S}_k^r} \sum_{j=0}^{2} h^j \| D^j(v - x) \|_{L^p(I)} \leqq Ch^s \| v \|_{W_p^s(I)}. \tag{4–2}$$

A natural Galerkin approximation to (2–1) is: find $u^h : [0, T] \to \mathring{S}_k^r$ such that

$$\left(\frac{\partial u^h}{\partial t}, \chi\right) + \gamma(D^2 u^h, D^2\chi) = (\varphi(u^h), D^2\chi) \quad \forall\, \chi \in \mathring{S}_k^r \tag{4–3a}$$

$$u^h(0) = u_0^h \tag{4–3b}$$

where $u_0^h \in \mathring{S}_k^r$ is a suitable approximation to $u_0$. We note that since $\chi = 1$ belongs to $\mathring{S}_k^r$, (4–3a) implies

$$\frac{1}{L}(u^h(t), 1) = \frac{1}{L}(u_0^h, 1). \tag{4–4}$$

The global existence theorems of section 2 can be extended to the Galerkin approximation (4-3).

**Proposition 4.1.**

(*a*) *If* $\gamma_2 > 0$ *then for any initial data* $u_0^h \in \mathring{S}_k^r$ *and* $T > 0$ *there exists a unique global solution* $u^h \in H^{2,1}(Q_T)$ *to* (4.3).

(*b*) *If* $\gamma > L^2/\pi^2$ *and the initial data* $u_0^h \in \mathring{S}_k^r$ *is such that* $\|u_0^h\|_2$ *is sufficiently small, then there exists a unique global solution* $u^h \in H^{2,1}(Q_T)$ *to* (4–3).

**Proof.** Local existence and uniqueness is proved using Picard iteration. Global existence will follow from *a priori* bounds.

(a) Taking $\chi = u^h(t)$ in (4–3a) leads to the estimates, as in the derivation of (2–6),

$$\|u^h(t)\| \leqq C_T \|u_0^h\|, \tag{4–5a}$$

$$\int_0^t \|D^2u^h(\tau)\|^2 \, d\tau \leqq C_T \|u_0^h\|^2. \tag{4–5b}$$

Since $\mathring{S}_k^r$ is a finite-dimensional space (4–5a) also implies that, for fixed $h$, $\|u^h(t)\|_\infty$ is uniformly bounded on $[0, T]$ which is sufficient to deduce global existence for the ordinary differential equations (4–3) since $\varphi(\cdot)$ is continuously differentiable.

(b) Setting

$$v^h(t) = u^h(t) - \frac{1}{L}(u_0^h, 1) \tag{4–6}$$

and following the arguments leading up to inequality (2.47) of Theorem 2.2 yields the desired assertion. □

Associated with $\mathring{S}_k^r$ is the elliptic projection $P^h : H_E^2(I) \to \mathring{S}_k^r$ defined by: for $v \in H_E^2(I)$ then $P^h v$ satisfies

$$(D^2P^hv - D^2v, D^2\chi) = 0 \quad \forall\, \chi \in \mathring{S}_k^r \text{ and } (\chi, 1) = 0, \tag{4–7a}$$

$$(P^hv - v, 1) = 0. \tag{4–7b}$$

The existence of a unique $P^hv$ satisfying (4-7) follows from the Lax-Milgram theorem and the Friedrichs-Poincaré inequality

$$\|\eta\|_2 \leqq C\{|\eta|_2 + |(\eta, 1)|\}, \quad \forall\, \eta \in H_E^2(I). \tag{4–8}$$

**Theorem 4.1.** *Suppose that the solution $u(t)$ of (2–1) is sufficiently regular for a given $T > 0$ and that the solution of (4–3) satisfies*

$$\|u^h(t)\|_\infty \leqq C_T, \quad 0 \leqq t \leqq T. \tag{4–9}$$

*If the initial data satisfy*

$$\|u_0 - u_0^h\| \leqq Ch^r \quad \text{and} \quad (u_0^h, 1) = M, \tag{4–10}$$

*then*

$$t^{\frac{1}{4}} \|u(t) - u^h(t)\|_\infty + \|u(t) - u^h(t)\| \leqq C_T(u)\, h^r \quad \forall\, t \in (0, T]. \tag{4–11}$$

*If $u_0^h = P^h u_0$ then*

$$\sup_{t \in (0,T)} \sum_{j=0}^{2} h^j |u(t) - u^h(t)|_j \leqq C_T(u)\, h^r, \tag{4–12a}$$

$$\left\| \frac{\partial u}{\partial t} - \frac{\partial u^h}{\partial t} \right\|_{L^2(0,T;L^2(I))} \leqq C_T(u)\, h^r, \tag{4–12b}$$

$$\|u(t) - u^h(t)\|_\infty \leqq C_T(u)\, h^r \quad \forall\, t \in [0, T]. \tag{4–12c}$$

**Proof.** Our method of proof is based on the error decomposition

$$u - u^h = \theta^h + e^h, \quad \theta^h \equiv u - P^h u, \quad e^h \equiv P^h u - u^h \tag{4–13}$$

(*cf.* WHEELER [1973], THOMÉE [1974] and WAHLBIN [1975] for linear parabolic equations) and the following proposition regarding the projection $P^h$.

**Proposition 4.2.** *For*

$$v \in H_E^2(I) \cap H^r(I),$$

$$\sum_{j=0}^{2} h^j |v - P^h v|_j \leqq C h^r \|v\|_r \tag{4–14a}$$

*and if* $v \in H_E^2(I)$, *then*

$$\|v - P^h v\|_\infty \leqq C h^r \|v\|_{W_\infty^r(I)}. \quad \square \tag{4–14b}$$

We assume Proposition 4–2 for the moment and postpone its proof to the end of this section. It follows from (4–14) and the assumption concerning the regularity of $u$ that

$$\sup_{t \in (0,T)} \sum_{j=0}^{2} h^j |\theta^h(t)|_j \leqq C_T(u)\, h^r \tag{4–15a}$$

$$\left\| \frac{\partial \theta^h}{\partial t} \right\|_{L^2(0,T;L^2(2))} \leqq C_T(u)\, h^r \tag{4–15b}$$

$$\|\theta^h(t)\|_\infty \leqq C_T(u)\, h^r \quad 0 \leqq t \leqq T. \tag{4–15c}$$

We obtain (4–15b) by applying proposition 4.2 with $v = \partial u / \partial t$.

Hence it remains to obtain the corresponding appropriate bounds for $e^h$. Observe that, by (4–7a) and (4–3a), for all $\chi \in \mathring{S}_k^r$ and $(\chi, 1) = 0$

$$\left(\frac{\partial e^h}{\partial t}, \chi\right) + \gamma(D^2 e^h, D^2 \chi) = \left(-\frac{\partial \theta^h}{\partial t}, \chi\right) + (\varphi(u) - \varphi(u^h), D^2 \chi). \tag{4–16}$$

Taking $\chi = e^h$ in (4–16) we obtain the inequality

$$\tfrac{1}{2} \frac{d}{dt} \|e^h\|^2 + \gamma |e^h|_2^2 \leqq \left\| \frac{\partial \theta^h}{\partial t} \right\| \|e^h\| + C \|u - u^h\| \, |e^h|_2 \tag{4–17}$$

where the continuous differentiability of $\varphi(\cdot)$ and the *a priori* $L^\infty$ bounds on $u$ and $u^h$ have been used. It follows from (4–17) that

$$\tfrac{1}{2} \frac{d}{dt} \|e^h\|^2 + \frac{\gamma}{2} |e^h|_2^2 \leqq C \left\{ \|e^h\|^2 + \|\theta^h\|^2 + \left\| \frac{\partial \theta^h}{\partial t} \right\|^2 \right\}$$

and by Gronwall's inequality that

$$\begin{aligned} \|e^h(t)\|^2 + \int_0^t |e^h(\tau)|_2^2 \, d\tau &\leqq \|e^h(0)\|^2 + C_T(u)\, h^{2r} \\ &\leqq C_T(u)\, h^{2r} \end{aligned} \tag{4–18}$$

where we have used (4–15a, b) and the observation that

$$\|e^h(0)\| \leqq \|u_0 - u_0^h\| + \|P^h u_0 - u_0\|$$

with (4–10) and (4–14a) holding. Of course in the case $u_0^h = P^h u_0$ we have that $e^h(0) = 0$.

Taking $\chi = \frac{\partial e^h}{\partial t}$ in (4–16), we obtain

$$\left\|\frac{\partial e^h}{\partial t}\right\|^2 + \frac{\gamma}{2}\frac{d}{dt}|e^h|_2^2 \leqq \left\|\frac{\partial \theta^h}{\partial t}\right\| \left\|\frac{\partial e^h}{\partial t}\right\| + \left(\varphi(u) - \varphi(u^h), D^2 \frac{\partial e^h}{\partial t}\right)$$

and after integrating with respect to $t$,

$$\begin{aligned} \tfrac{1}{2}\int_{t_1}^{t} \left\|\frac{\partial e^h}{\partial t}\right\|^2 d\tau + \frac{\gamma}{2}|e^h(t)|_2^2 &\leqq \frac{\gamma}{2}|e^h(t_1)|_2^2 + \tfrac{1}{2}\int_{t_1}^{t}\left\|\frac{\partial \theta^h}{\partial t}\right\|^2 d\tau \\ &+ \int_{t_1}^{t} \frac{d}{dt}\{(\varphi(u) - \varphi(u^h), D^2 e^h)\}\, d\tau \qquad (4\text{–}19) \\ &- \int_{t_1}^{t}\left(\varphi'(u)\frac{\partial u}{\partial t} - \varphi'(u^h)\frac{\partial u^h}{\partial t}, D^2 e^h\right) d\tau . \end{aligned}$$

Label the last two terms on the right-hand side of (4–19) as $I_1$, and $I_2$. Then using the boundedness of $u^h$ and $u$,

$$\begin{aligned} |I_1| &\leqq C(\|e^h(t)\|^2 + \|\theta^h(t)\|^2 + \|e^h(t_1)\|^2 + \|\theta^h(t_1)\|^2) + \frac{\gamma}{4}(|e^h(t)|_2^2 + |e^h(t_1)|_2^2) \\ &\leqq C_T(u)\, h^{2r} + \frac{\gamma}{4}|e^h(t)|_2^2 + \frac{\gamma}{4}|e^h(t_1)|_2^2 \qquad (4\text{–}20\text{a}) \end{aligned}$$

where the bounds (4–18), (4–10) and (4–15a) have been used. Turning to $I_2$ we find that

$$\begin{aligned} |I_2| &\leqq \int_0^t \left\{\left\|(\varphi'(u) - \varphi'(u^h))\frac{\partial u}{\partial t}\right\| + \left\|\varphi'(u^h)\frac{\partial e^h}{\partial t}\right\| + \left\|\varphi'(u^h)\frac{\partial \theta^h}{\partial t}\right\|\right\}|e^h|_2\, d\tau \\ &\leqq \tfrac{1}{4}\int_0^t \left\|\frac{\partial e^h}{\partial t}\right\|^2 d\tau + C_T(u)\int_0^t \left\{\|e^h\|^2 + \|\theta^h\|^2 + \left\|\frac{\partial \theta^h}{\partial t}\right\|^2 + |e^h|_2^2\right\} d\tau \\ &\leqq \tfrac{1}{4}\int_0^t \left\|\frac{\partial e^h}{\partial t}\right\|^2 d\tau + C_T(u)\, h^{2r} \qquad (4\text{–}20\text{b}) \end{aligned}$$

where we have used the differentiability of $\varphi(\cdot)$, the boundedness of $u^h$ and $\partial u/\partial t$ and the error bounds (4–18), (4–10) and (4–15). It follows from (4–19) and (4–20) that

$$\int_{t_1}^{t} \left\| \frac{\partial e^h}{\partial t} \right\|^2 d\tau + \gamma \, |e^h(t)|_2^2 \leqq \gamma \, |e^h(t_1)|_2^2 + C_T(u) \, h^{2r}. \tag{4–21}$$

In the case $u_0^h = P^h u_0$ we have from (4–18) and (4–21) (taking $t_1 = 0$) that

$$\|e^h(t)\|^2 + |e^h(t)|_2^2 + \int_0^t \left\| \frac{\partial e^h}{\partial t} \right\|^2 d\tau \leqq C_T(u) \, h^{2r}. \tag{4–22}$$

Otherwise (4–21) and (4–18) imply that

$$\begin{aligned} \gamma t \, |e^h(t)|_2^2 &\leqq \gamma \int_0^t \{|e^h(\tau)|_2^2 + C_T(u) \, h^{2r}\} \, d\tau \\ &\leqq \gamma C_T(u) \, h^{2r}. \end{aligned} \tag{4–23}$$

Since

$$|e^h(t)|_1^2 \leqq |e^h(t)|_2 \, \|e^h(t)\|$$

and

$$\|e^h(t)\|_\infty \leqq C \, \|e^h(t)\|_1,$$

it follows from (4–18), (4–22), (4–23) and (4–15) that (4–11) and (4–12) hold.

**Proof of Proposition 4.2.** The projection property of $P^h$ yields

$$|v - P^h v|_2 \leqq \inf_{\substack{\chi \in \mathring{S}_k^r \\ (\chi - v, 1) = 0}} |v - \chi|_2$$

and noting that

$$D^2(\chi - (\chi, 1) + (v, 1) \, L - v) = D^2(\chi - v), \tag{4–24}$$

together with the approximation (4–2) ($p = 2$, $s = t$) we obtain

$$|v - P^h v|_2 \leqq C h^{r-2} \, \|v\|_r. \tag{4–25}$$

The $L^2$-norm of the error is bounded by use of the usual duality argument. For any $\eta \in L^2(I)$, let $z \in H_E^2(I)$ be the unique solution of

$$(D^2 z, D^2 \xi) = (\eta, \xi) \qquad \forall \, \xi \in H_E^2(I), \quad (\xi, 1) = 0 \tag{4–26a}$$

$$(z, 1) = 0. \tag{4–26b}$$

It follows from (4–26) that

$$\|z\|_4 \leqq C(|z|_2 + |z|_4) \leqq C \, \|\eta\|. \tag{4–27}$$

Equations (4–7a) and (4–26a) yield

$$\begin{aligned} (v - P^h v, \eta) &= (D^2(v - P^h v), D^2 z) \\ &= (D^2(v - P^h v), D^2(z - \chi)) \quad \forall \, \chi \in \mathring{S}_k^r \quad \text{and} \quad (\chi, 1) = 0 \end{aligned}$$

and, hence using (4–24), (4–27) and (4–2) we obtain

$$\begin{aligned}(v - P^h v, \eta) &\leqq |v - P^h v|_2 \, |z - \chi|_2 \\ &\leqq Ch^r \|v\|_r \|z\|_4 \\ &\leqq Ch^r \|v\|_r \|\eta\|,\end{aligned}$$

so that

$$\|v - P^h v\| \leqq Ch^r \|v\|_r. \tag{4–28}$$

Therefore, noting the inequality (1–11), we have proved (4–14a).
It remains to prove the $L^\infty$ bound. First observe that by (4–7a)

$$(D^2 P^h v - D^2 v, \eta) = 0 \quad \forall \eta \in S_{k-2}^{r-2}, \quad (\eta, 1) = 0 \tag{4–29}$$

and since

$$(D^2 P^h v - D^2 v, 1) = 0$$

we have that $D^2 P^h v$ is the $L^2$ projection of $D^2 v$ in $S_{k-2}^{r-2}$. It follows from the $L^\infty$ error bound for the $L^2$ projection, due to DOUGLAS, DUPONT & WAHLBIN [1975], that

$$\|D^2(v - P^h v)\|_\infty \leqq Ch^{r-2} \|D^2 v\|_{W_\infty^{r-2}(I)}. \tag{4–30}$$

Using the dual problem (4–26) with $\eta \in L^1(I)$ so that

$$\|z\|_{W_1^4(I)} \leqq C \|\eta\|_{L^1(I)}, \tag{4–31}$$

we have

$$\begin{aligned}(v - P^h v, \eta) &= (D^2(v - P^h v), D^2 z) \\ &= (D^2(v - P^h v), D^2(z - \chi)) \\ &\leqq \|D^2(v - P^h v)\|_\infty \|D^2(z - \chi)\|_{L^1(I)} \quad \forall \chi \in \mathring{S}_{k'}^{r}\, (\chi, 1) = 0.\end{aligned} \tag{4–32}$$

It follows from (4–24), (4–2) with $p = 1$ and $s = 4$, (4–30), (4–31) and (4–32) that

$$\|v - P^h v\|_\infty \leqq Ch^r \|v\|_{W_\infty^r(I)}. \quad \square$$

*Remarks.* 1. The assumption (4–9) is not a restriction. By a standard argument (see THOMÉE [1984; p. 154]) we may use the error bounds (4–11) or (4–12) in order to justify (4–9) *a posteriori* for any $T > 0$ such that (2–1) has a solution.
2. The smoothing property of the linearized differential operator is responsible for the $L^\infty$ error bound in (4–11) for any $t > 0$ despite there being no assumption on the initial $L^\infty$ error.

*Acknowledgement.* This work was partially supported by the National Science Foundation, Grant No. DMS-8501397 and the Air Force Office of Scientific Research and the Office of Naval Research. ZHENG expresses his thanks to J. SHATAH and F. BERNIS for their very helpful discussions.

## References

R. A. ADAMS, *Sobolev Spaces*, Academic Press, New York (1975).

J. W. CAHN & J. E. HILLIARD, Free energy of a nonuniform system. I. Interfacial free energy, *J. Chem. Phys.* **28** (1958), 258–267.

J. CARR, M. E. GURTIN & M. SLEMROD, Structural phase transitions on a finite interval, *Arch. Rational Mech. Anal.* **12** (1984), 317–351.

D. S. COHEN & J. D. MURRAY, A generalized diffusion model for growth and dispersal in a population, *J. Math. Biology* **12** (1981), 237–249.

J. DOUGLAS, Jr., T. DUPONT & L. WAHLBIN, Optimal $L^\infty$ error estimates for Galerkin approximations to solutions of two point boundary value problems, *Math. Comp.* **29** (1975), 475–483.

M. HAZEWINKEL, J. F. KAASHOEK & B. LEYNSE, *Pattern formation for a one dimensional evolution equation based on Thom's river basin model*, Report # 8519/B, Econometric Institute, Erasmus University (1985).

S. KLAINERMAN & G. PONCE, Global small amplitude solutions to nonlinear evolution equations, *Comm. Pure. Appl. Math.* **36** (1983), 133–141.

J. L. LIONS & E. MAGENES, *Non-homogeneous boundary value problems and applications*, Vol. II, Springer-Verlag (1972).

A. NOVICK-COHEN, *Energy methods for the Cahn-Hilliard equation*, IMA Preprint # 157, (1985).

A. NOVICK-COHEN & L. A. SEGEL, Nonlinear aspects of the Cahn-Hilliard equation, *Physica* **10** (D) (1984), 277–298.

G. I. SIVASHINSKY, On cellular instability in the solidification of a dilute binary alloy, *Physica* **8** (D) (1983), 243–248.

V. THOMÉE, Some convergence results for Galerkin methods for parabolic boundary value problems, in *Mathematical Aspects of Finite Element Methods in Partial Differential Equations*, ed. C. de BOOR, Academic Press (1974), p. 55–84.

V. THOMÉE, *Galerkin Finite Element Methods for Parabolic Problems*, L. N. M. # 1054, Springer-Verlag, Berlin (1984).

L. WAHLBIN, On maximum norm error estimates for Galerkin approximations to one dimensional second order parabolic boundary value problems, *SIAM J. Numer. Anal.* **12** (1975), 177–187.

M. F. WHEELER, A priori $L^2$ error estimates for Galerkin approximations to parabolic partial differential equations, *SIAM J. Numer. Anal.* **10** (1973), 723–759.

Department of Mathematics
Purdue University
West Lafayette, Indiana

*(Received March 17, 1986)*