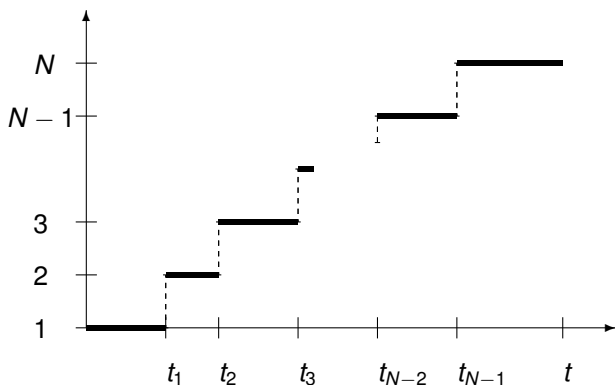


# Directed polymers and the quantum Toda lattice

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# A directed polymer model



A path  $\phi \equiv \{0 < t_1 < \dots < t_{N-1} < t\}$ .

# A directed polymer model

The environment:  $B_1, B_2, \dots$  independent standard 1-dim Brownian motions.

For  $\phi \equiv \{0 < t_1 < \dots < t_{N-1} < t\}$ , define

$$E(\phi) = B_1(t_1) + B_2(t_2) - B_2(t_1) + \dots + B_N(t) - B_N(t_{N-1}).$$

Boltzmann measure:

$$\mathbb{P}(d\phi) = Z_t^N(\beta)^{-1} e^{\beta E(\phi)} d\phi, \quad Z_t^N(\beta) = \int e^{\beta E(\phi)} d\phi.$$

# The free energy density

Theorem (O'C-Yor '01, O'C-Moriarty '07)

*Almost surely,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log Z_N^N(\beta) = \inf_{t > 0} [t\beta^2 - \Psi(t)] - \log \beta^2 =: f(\beta),$$

where  $\Psi(z) = \Gamma'(z)/\Gamma(z)$ .

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For small  $\beta$ ,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \frac{Z_N^N(\beta)}{\mathbb{E} Z_N^N(\beta)} \sim \frac{5}{24} \beta^4.$$

# Fluctuations

## Theorem (Seppalainen-Valko 2010)

*There exist finite, positive  $\beta$ -dependent constants  $C$ ,  $b_0$  and  $N_0$  such that for  $b \geq b_0$  and  $N \geq N_0$ ,*

$$\mathbb{P} \left( \left| \log Z_N^N(\beta) - f(\beta)N \right| \geq bN^{1/3} \right) \leq Cb^{-3/2}.$$

# A scaling property

By Brownian scaling,

$$(Z_t^N(\beta), t \geq 0) \stackrel{d}{=} (\beta^{-2(N-1)} Z_{\beta^2 t}^N(1), t \geq 0).$$

Define

$$Z_t^N = Z_t^N(1).$$

## As an interacting particle system ...

Set  $X_t^N = \log Z_t^N$ . Then

$$dX_t^N = e^{X_t^{N-1} - X_t^N} dt + dB_t^N.$$

Infinite system has product-form invariant measure for each given intensity.

This allows computation of the free energy density following Rost (1986) / Seppalainen (1998), analogous to TASEP.

## Connection with random matrices

The law of  $Z_t^N(\beta)$  is well-understood in the zero temperature limit. Define

$$\begin{aligned} M_t^N &= \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log Z_t^N(\beta) \\ &= \max_{0=t_0 \leq t_1 \leq \dots \leq t_{N-1} \leq t_N=t} \sum_{i=1}^N B_i(t_i) - B_i(t_{i-1}). \end{aligned}$$

The process  $(M_t^N, t \geq 0)$  is  $B^N$  'reflected off'  $B^{N-1}$  'reflected off' ... 'reflected off'  $B^2$  'reflected off'  $B^1$ .

By Brownian scaling, the law of  $M_t^N/\sqrt{t}$  is independent of  $t$ .

## Connection with random matrices

Theorem (Baryshnikov '01, Gravner-Tracy-Widom '01)

*The random variable  $M_1^N$  has the same law as the largest eigenvalue of a  $N \times N$  GUE random matrix, that is*

$$\mathbb{P}(M_1^N \leq y) = \int_{\max_{1 \leq i \leq N} x_i \leq y} c_N e^{-\sum_{i=1}^N x_i^2/2} h(x)^2 dx$$

where

$$h(x) = \prod_{1 \leq i < j \leq N} (x_i - x_j)$$

and  $c_N$  is a normalisation constant.

## Connection with random matrices

This yields very precise information about the law and asymptotic behavior of  $M^N$ . For example,

$$P(M_N^N \leq y) = \det[1 - K_N]_{L_2([y, \infty))}$$

where  $K_N$  is the ‘Hermite kernel’, and

$$\lim_{N \rightarrow \infty} P\left(M_N^N \leq 2N + yN^{1/3}\right) = F_2(y),$$

where

$$F_2(y) = \det[1 - K_{\text{Airy}}]_{L_2([y, \infty))}$$

is the Tracy-Widom distribution.

## Connection with random matrices

In fact [Bougerol-Jeulin '02, O'C-Yor '02] the stochastic process  $(M_t^N, t \geq 0)$  has the same law as the top line of a system of  $N$  Dyson Brownian motions. That is, it has the same law as the first coordinate of a Brownian motion conditioned never to exit

$$C_N = \{x \in \mathbb{R}^N : x_1 > \dots > x_N\},$$

started from the origin. This is a diffusion in  $\overline{C}_N$  with generator

$$\frac{1}{2}h(x)^{-1} \Delta_{C_N} h(x) = \Delta/2 + \nabla \log h \cdot \nabla.$$

# The quantum Toda lattice

The quantum Toda lattice is a quantum integrable system with Hamiltonian given by

$$H = \Delta - 2 \sum_{i=1}^{N-1} e^{x_{i+1} - x_i}.$$

The eigenfunctions  $\psi_\lambda$  of  $H$  are naturally indexed by  $\lambda \in \iota\mathbb{R}^N$ , given explicitly by an integral formula due to Givental (1997).

There is a positive eigenfunction  $\psi_0$  with  $H\psi_0 = 0$ .

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Funny fact:  $\psi_0(x)$  is the ‘volume’ of the set of ‘Gelfand-Tsetlin patterns’ with top row  $x$ , but with indicator functions  $1_{a \leq b}$  replaced by double exponentials  $\exp(-e^{a-b})$ .

# The process $Z_t^N$

## Theorem

*The stochastic process  $\log Z_t^N$ ,  $t > 0$  has the same law as the \*first coordinate\* of the diffusion in  $\mathbb{R}^N$  with generator*

$$\mathcal{L} = \frac{1}{2} \psi_0^{-1} H \psi_0 = \frac{1}{2} \Delta + \nabla \log \psi_0 \cdot \nabla$$

*started from  $-\infty$ .*

This diffusion can be thought of as a geometric analogue of Dyson's Brownian motion.

## The other coordinates

Set  $X_1^N(t) = \log Z_t^N$  and, for  $k = 2, \dots, N$ ,

$$X_1^N(t) + \dots + X_k^N(t) = \log \int e^{E(\phi_1) + \dots + E(\phi_k)} d\phi_1 \dots d\phi_k,$$

where the integral is over non-intersecting paths  $\phi_1, \dots, \phi_k$  from  $(0, 1), \dots, (0, k)$  to  $(t, N - k + 1), \dots, (t, N)$ .

### Theorem

*The process  $X^N$  is a diffusion process in  $\mathbb{R}^N$  with generator  $\mathcal{L}$ .*

cf. Greene's theorem: this is based on a geometric variant of the RSK correspondence (cf. Kirillov 2000).

Generalizes a theorem of Matsumoto and Yor (1999), which in turn is a geometric analogue of Pitman's '2M - X' theorem.

# Proof uses theory of Markov functions

Set  $X^1 = B^1$ . It is easy to see that  $(X^1, \dots, X^N)$  is a Markov process in  $\mathbb{R} \times \mathbb{R}^2 \dots \times \mathbb{R}^N$  which satisfies a simple SDE. The Markov property of  $X^N$  follows from an intertwining relation plus some technical results concerning the entrance from  $-\infty$ .

# The entrance law

The entrance law  $\mu_t$  from  $-\infty$  is given by

$$\mu_t(dx) = \psi_0(x) \int_{\mathcal{L}\mathbb{R}^N} \exp\left(\frac{1}{2} \sum_{i=1}^N \lambda_i^2 t\right) \psi_\lambda(x) s_N(\lambda) d\lambda,$$

where

$$s_N(\lambda) = \frac{1}{(2\pi\iota)^N N!} \prod_{j \neq k} \Gamma(\lambda_j - \lambda_k)^{-1}$$

is the *Sklyanin measure* - the Plancherel measure for the quantum Toda lattice [Sklyanin 1985, Semenov-Tian-Shanski 1994, Kharchev-Lebedev 1999].

The measure  $\mu_t(dx)$  is a ‘deformation’ of the GUE.

# The law of the partition function

## Corollary

For  $s > 0$ ,

$$Ee^{-sZ_t^N} = \int s^{\sum \lambda_i} \prod_i \Gamma(-\lambda_i)^N e^{\frac{1}{2} \sum_i \lambda_i^2 t} s_N(\lambda) d\lambda,$$

where the integral is along vertical lines with  $\Re \lambda_i < 0$  for all  $i$ .

This uses a remarkable identity, conjectured by Bump (1989), proved by Stade (2002), and extended / elucidated in the present context by Gerasimov, Lebedev and Oblazin (2008). Moreover, the RHS is a Fredholm determinant.

## Connection with random matrices

The probability measure on  $\ell\mathbb{R}^N$  with density proportional to

$$e^{\sum_i \lambda_i^2 t/2} s_N(\lambda) \equiv \frac{1}{(2\pi\ell)^N N!} e^{\sum_i \lambda_i^2 t/2} \prod_{i>j} (\lambda_i - \lambda_j) \prod_{i<j} \frac{\sin \pi(\lambda_i - \lambda_j)}{\pi}$$

is (up to a factor of  $\ell\pi$ ) the law, at time  $1/t$ , of the radial part of a Brownian motion in the symmetric space of positive definite Hermitian matrices. In particular, it is a determinantal point process, so  $Ee^{-sZ_t^N}$  can be written as a Fredholm determinant.

# Crossover distributions

The law of  $\log Z_t^N$  should converge (in an appropriate scaling) to the 'crossover distributions' recently introduced in the context of the KPZ / stochastic heat equation by Sassamoto-Spohn (2010) and Amir-Corwin-Quastel (2010) - building on recent work of Tracy and Widom on ASEP - and also via a different approach by Dotsenko-Klumov (2010).

The above RSK-type construction extends naturally to the continuum setting.