Tropical combinatorics and Whittaker functions

Neil O'Connell, University of Warwick

Limit Theorems in Probability, Statistics and Number Theory
Conference in honor of Friedrich Götze's 60th birthday
Bielefeld, August 2011

Joint with Ivan Corwin, Timo Seppäläinen, Nikos Zygouras

Schur polynomials

An integer partition is a sequence of natural numbers $\lambda_1 \geq \lambda_2 \geq \ldots$ such that $\sum_i \lambda_i < \infty$.

For λ a partition, define

$$a_\lambda(x_1, \ldots, x_n) = \det[x_i^{\lambda_j}]_{i,j=1}^n.$$

For $\delta = (n-1, n-2, \ldots, 1, 0)$,

$$a_\delta(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

Schur polynomials

The Schur polynomial in n variables corresponding to the partition λ is defined by

$$s_\lambda(x_1,\dots,x_n) = a_{\delta+\lambda}(x_1,\dots,x_n)/a_\delta(x_1,\dots,x_n).$$

The polynomials s_λ are symmetric and homogeneous with positive integer coefficients.

E.g.

$$s_{(2,1,1)}(x_1,x_2,x_3) = x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2.$$

Cauchy's identity

For $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$, we have

$$\sum_{\lambda} s_\lambda(x) s_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1},$$

where the sum is over partitions λ with at most $n \wedge m$ parts.

Schur measures on partitions

If $p = (p_1, \ldots, p_m)$ and $q = (q_1, \ldots, q_n)$ are all in $(0, 1)$, then

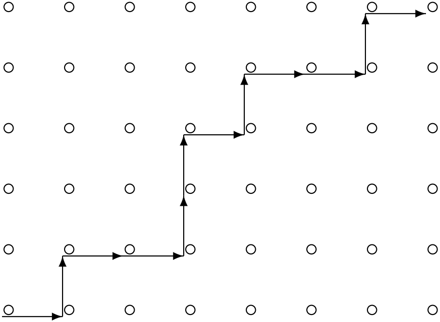
$$\mu_{p,q}(A) = \prod_{i,j}(1 - p_i q_j) \sum_{\lambda \in A} s_\lambda(p) s_\lambda(q)$$

defines a probability measure on the set of partitions with at most $m \wedge n$ parts. These are known as Schur measures.

The RSK correspondence

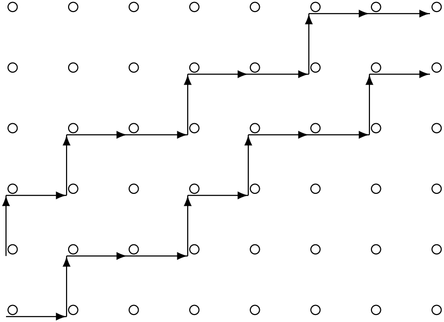
There is a combinatorial construction in the theory of Young tableaux known as the RSK (Robinson-Schensted-Knuth) correspondence, which provides a realisation of a Schur-distributed random partition in terms of independent geometric random variables and non-intersecting lattice paths. (It also gives a combinatorial proof of Cauchy's identity.)

Non-intersecting lattice paths



A lattice path in $\Pi^1_{8,6}$

Non-intersecting lattice paths



A pair of non-intersecting lattice paths in $\Pi^2_{8,6}$

A Schur-distributed random partition

Let w_{ij} be independent geometric random variables with respective paramaters $p_i q_j$.

Define $L_1 \geq L_2 \geq \cdots \geq L_{m\wedge n}$ by

$$L_1 + \cdots + L_k = \max_{\Gamma \in \Pi^k_{m,n}} \sum_{(i,j)\in\Gamma} w_{ij}.$$

Then L is distributed according to the Schur measure $\mu_{p,q}$.

Tropical RSK

A.N. Kirillov (2000) introduced a ‘geometric’ analogue of the RSK correspondence, which is defined by replacing $(\max, +)$ by $(+, \times)$.

In this setting we define

$$Y_1 \cdots Y_k = \sum_{\Gamma \in \Pi^k_{m,n}} \prod_{(i,j)\in\Gamma} w_{ij}.$$

Tropical RSK with random input

Theorem

If w_{ij} are independent inverse gamma random variables with respective parameters $\theta_i + \hat{\theta}_j$, and suppose $m \geq n$. Then the distribution of $Y = (Y_1, \ldots, Y_n)$ is characterized by

$$\int_{\mathbb{R}^n} W_{\theta,\lambda}(y) \mathbb{P}(Y \in dy) = \prod_{i,j} \frac{\Gamma(\theta_i + \hat{\theta}_j + \lambda_i)}{\Gamma(\theta_i + \hat{\theta}_j)},$$

where $W_{\theta,\lambda}$ are (normalized) $GL(n, \mathbb{R})$-Whittaker functions.

The law of the partition function Y_1

Corollary

For $s > 0$,

$$Ee^{-sY_1} = \int s^{\sum(\theta_i - \lambda_i)} \prod_{i,j} \Gamma(\lambda_i - \theta_j) \prod_{i,j} \frac{\Gamma(\lambda_i + \hat{\theta}_j)}{\Gamma(\theta_i + \hat{\theta}_j)} s_N(\lambda) d\lambda,$$

where

$$s_N(\lambda) = \frac{1}{(2\pi\iota)^N N!} \prod_{j \neq k} \Gamma(\lambda_j - \lambda_k)^{-1}$$

and the integral is along vertical lines with $\Re\lambda_i > \theta_j$ for all i, j.

Bump-Stade Whittaker integral identity

The corollary is deduced using an ‘integrating out lemma’ for $GL(n,\mathbb{R})$-Whittaker functions due to Bump (1989) and Stade (2002). If we write $W_{\theta,\lambda} = \psi_{\theta+\lambda}/\psi_{\theta}$, then

$$\int_{\mathbb{R}_+^n} e^{-sy_1} \psi_\lambda(y)\psi_\nu(y) \prod_{i=1}^{N} \frac{dy_i}{y_i} = s^{\sum(\lambda_i+\nu_i)} \prod_{i,j} \Gamma(-\lambda_i-\nu_j).$$

Determinantal structure

The ‘Sklyanin measure’

$$s_N(\lambda) = \frac{1}{(2\pi\iota)^N N!} \prod_{i>j}(\lambda_i - \lambda_j) \prod_{i<j} \frac{\sin \pi(\lambda_i - \lambda_j)}{\pi}.$$

is a product of two determinants. In particular, Ee^{-sY_1} can be written as a Fredholm determinant.

Remarks

The random variable Y_1 is the partition function of a discrete directed polymer model introduced by Seppäläinen (2009).

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A Brownian version of this model was considered in O'C-Yor (2001), Moriarty-O'C (2007) and O'C (2009), which gives rise to a positive temperature version of the Gaussian unitary ensemble and is closely related to the quantum Toda lattice; the Brownian model can be regarded as a particular scaling limit of the above discrete model and also features $GL(n, \mathbb{R})$-Whittaker functions in an essential way.

Positive temperature ensembles

LUE:

$$\int_{\mathbb{R}^n} W_{\theta,\lambda}(y)\mu(dy) = \prod_i \left[\frac{\Gamma(\theta+\lambda_i)}{\Gamma(\theta)}\right]^m,$$

GUE:

$$\int_{\mathbb{R}^n} W_{0,\lambda}(y)\mu(dy) = e^{\sum_i \lambda_i^2/2}$$

Dynamical aspects

The sequence of random partitions

$$L(m) = (L_1, L_2, \dots, L_n)(m) \qquad m = 1, 2, \dots$$

is a Markov chain, called a Schur process.

To prove the main result, we show, using the theory of Markov functions, that the sequence

$$Y(m) = (Y_1, \dots, Y_n)(m), \qquad m = 1, 2, \dots$$

is a Markov chain, the eigenfunctions of which are given by the normalized Whittaker functions $W_{\theta,\lambda}$.

Queueing interpretation of RSK

The basic building block which describes the dynamics of the RSK algorithm is the Lindley recursion:

$$v_{m+1} = \max\{v_m - a_m + s_{m+1}, s_{m+1}\}$$

$$a_m = \text{inter-arrival time between customers } m \text{ and } m+1$$

$$s_m = \text{service time of customer } m$$

$$v_m = \text{sojourn time of customer } m$$

The ‘output’:

$$d_m = \text{inter-departure time between customers } m \text{ and } m+1$$

$$r_m = \text{‘backward service time’ of customer } m$$

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$s_m =$ service time of customer m

$v_m =$ sojourn time of customer m

The ‘output’:

$$d_m = a_m - v_m + v_{m+1}$$

$$r_m = v_m - v_{m+1} + s_{m+1}$$

Output theorem for the M/M/1 queue

If $a_m, m \in \mathbb{Z}$ are iid exponential (geometric) and $s_m, m \in \mathbb{Z}$ are iid exponential (geometric), indep. of a, with $Es_0 < Ea_0$, then

$$v_{m+1} = \max\{v_m - a_m + s_{m+1}, s_{m+1}\}, \qquad m \in \mathbb{Z}$$

has a stationary solution and the output (d, r) has the same law as the input (a, s).

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has a stationary solution and the output (d, r) has the same law as the input (a, s).

The fact that RSK with independent geometric (or exponential) random variables as input is 'solvable' can be seen as a consequence of this output theorem.

Tropical M/M/1 queue

$$v_{m+1} = v_m a_m^{-1} s_{m+1} + s_{m+1}, \qquad m \in \mathbb{Z} \qquad (*)$$

$$d_m = a_m v_m^{-1} v_{m+1}, \qquad r_m = v_m v_{m+1}^{-1} s_{m+1}$$

Theorem (Seppalainen 2009)

If $a_m, m \in \mathbb{Z}$ are iid $\Gamma^{-1}(\theta)$ and $s_m, m \in \mathbb{Z}$ are iid $\Gamma^{-1}(\mu)$, indep. of a, with $\theta < \mu$, then (*) has a stationary solution and the output (d, r) has the same law as the input (a, s).

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This follows from the fact (Lukacs 1955) that, for independent random variables X and Y, $X/(X+Y)$ is independent of $X+Y$ iff X and Y are gamma distributed (up to a constant multiple).

Concluding remarks

Tropical RSK is closely related to Dodgson's 'condensation method' for evaluating determinants:

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The tropical RSK construction extends to the continuum setting of KPZ / stochastic heat equation, where it can be used to define a positive temperature version of the multi-layer Airy process, see O'C-Warren (2011).