Lecture 21 - 24/03/23

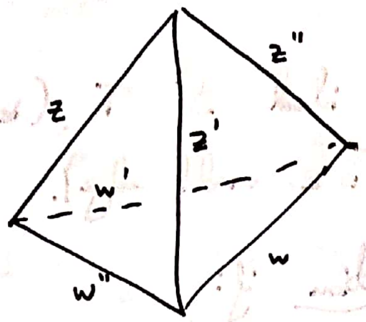
§4.7 - Shape Variety

Let M be a compact, connected, oriented 3-mfd w/ $\partial M \neq \emptyset$ and $\partial M = \sqcup T^2$. Fix, once and for all, an ideal triangulation T of M [ie. $|T| \setminus \{\text{vertices}\} \cong M \setminus \partial M$].

Defn 4.12: Define $\mathcal{S}(M,T) := \{ z : \{\text{modeled edges}\} \longrightarrow \mathbb{C} : z$ satisfies the tet equs & the edge equs$\}$

This is the shape variety.

P.T.O. ⟶



$w = z, \quad w' = z', \quad w'' = z'',$

$$z' = -\frac{1}{z-1}, \quad z'' = \frac{-1}{z'-1}$$

five tetrahedron equations

$$\left.\prod_{[e_i]=e} z(e_i) = 1 \right\} \text{ edge equation}$$

Note: if we have t tets, $\exists$ $6t$ edge parameters & $6t$ equations.

Exercise: # edges = # tets (so # eqns is $5t + t = 6t$).

This gives an algebraic variety.

Rem. 4.2: Note that $zz'z''ww'w'' = 1$ for any tet. Hence the product of all edge parameters is 1, so the product of all edge equations is 1.

So: can deduce any one edge eqn from the rest.

Thus: $\dim_{\mathbb{C}}(\mathcal{S}) \geq 1$ [algebraic geometry] (as long as $\mathcal{S} \neq \emptyset$).

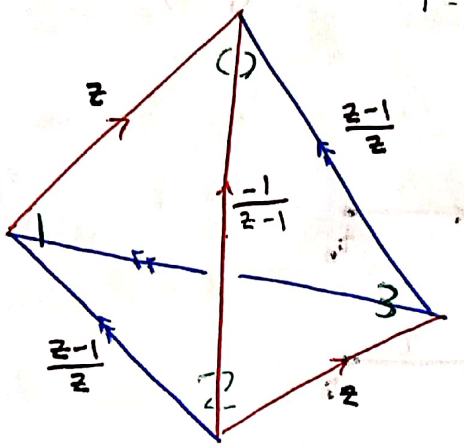
Exercise: $\dim_{\mathbb{C}}(\mathcal{S}) = |\partial M|$. [We will prove the case $|\partial M| = 1$ along the way of what we do today.]

Exercise: (i) Find an example of (M,T) with $\mathcal{S}(M,T) = \emptyset$.

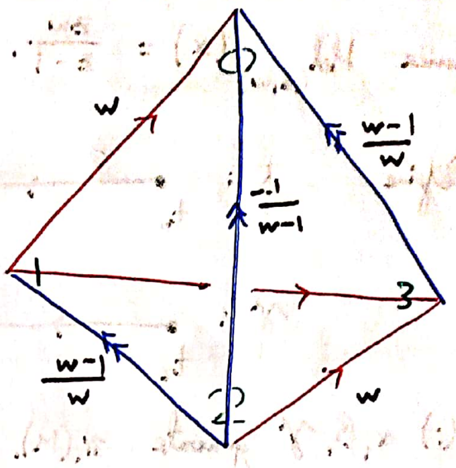
(ii) " " " " " where $\mathcal{S}(M,T) \neq \emptyset$ but contains no geometric structure.

Open: Suppose M is cusped, hyperbolic of finite volume. Does M admit a geometric ideal triangulation?

' = 'free' numbers



Tet 0, t_0



Tet 1, t_1

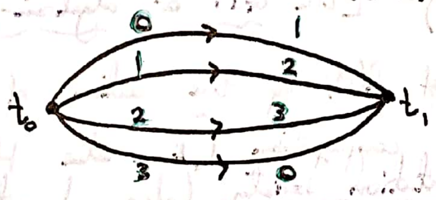
Calculate edge eqns:

$$z\left(\frac{-1}{z-1}\right) z\, w \left(\frac{-1}{w-1}\right) w = 1 \quad \text{ie. } z^2 w^2 = (z-1)(w-1)$$

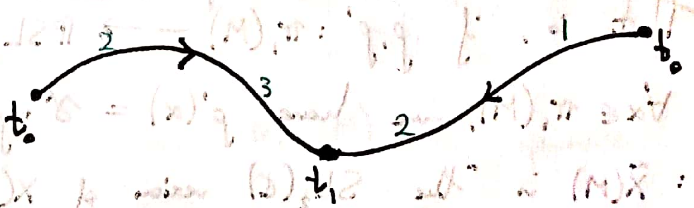
$$\left(\frac{\cancel{z-1}}{z}\right)\left(\frac{-1}{\cancel{z-1}}\right)\left(\frac{z-1}{z}\right)\left(\frac{\cancel{w-1}}{w}\right)\left(\frac{-1}{\cancel{w-1}}\right)\left(\frac{w-1}{w}\right) = 1 \quad \text{ie. } z^2 w^2 = (z-1)(w-1).$$

Break for movie night 1 - Snappy hyperbolic Δ^s.

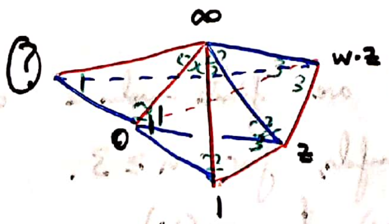
The dual 1-skeleton has two vertices + 4 edges:



Let α be the loop



We develop along α to get



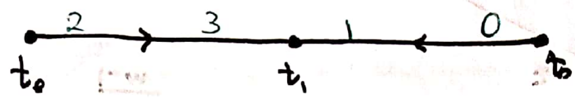
$$\rightsquigarrow \alpha: \left.\begin{array}{lll} 0: & \infty \longrightarrow & \infty \\ 1: & 0 \longmapsto & ? \\ 2: & 1 \longmapsto & 0 \\ 3: & z \longmapsto & w\cdot z \end{array}\right\} \text{call this } \rho_{(z,w)}(\alpha)$$

So $\rho_{(z,w)}(\alpha)(\zeta) = \dfrac{zw(\zeta-1)}{z-1}$ (using simple 3-transitivity of $PSL_2(\mathbb{C})$).

Hence $Hol_{(z,w)}(\alpha) = \dfrac{zw}{z-1}$.

Define β :

γ :



Exercise : (i) α, β, γ generate $\pi_1(M)$.

(ii) Compute the holonomies of β & γ.

(iii) In fact, $\pi_1(M) \cong \langle \alpha, \beta \mid \alpha\beta\alpha^{-1}\beta\alpha = \beta\alpha\beta^{-1}\alpha\beta \rangle$.

(iv) Do any other example.

§4.8 - Representation Varieties

Suppose M is as in §4.7, with $\partial M \cong \mathbb{T}^2$.

Def^n 4.13 : Define • $R(M) := Hom(\pi_1(M), PSL_2(\mathbb{C}))$

• $X(M) = R(M) /\!\!/ PSL_2(\mathbb{C})$ [the GIT or algebraic-geometric quotient]

i.e. the quotient of $R(M)$ by the action of $PSL_2(\mathbb{C})$ by postcomposition via conjugation [& make Hausdorff].

$X(M)$ is called the representation variety (aka. character variety).

That is, if $\rho, \rho' : \pi_1(M) \longrightarrow PSL_2(\mathbb{C})$ are rep^{n}s and $\exists\, \gamma \in PSL_2(\mathbb{C})$ st. $\forall \alpha \in \pi_1(M)$, we have $\rho'(\alpha) = \gamma \circ \rho(\alpha) \circ \gamma^{-1}$, then $[\rho] = [\rho']$ in $X(M)$.

Not^n : $\tilde{X}(M)$ is the $SL_2(\mathbb{C})$ version of $X(M)$.

Mostow-Prasad Rigidity

Th^m 4.3 Theorem (Mostow-Prasad Rigidity) : Suppose M, N are finite-volume complete hyperbolic n-mflds, for $n \geq 3$.

Suppose $\pi_1(M) \xrightarrow[\cong]{f} \pi_1(N)$.

Then $\exists$ isometry $\varphi : M \longrightarrow N$ st. $\boxed{\varphi_* = f}$

Rem. 4.3: The proof uses the conformal structure (that is, angles) in $\partial_\infty \mathbb{H}^n \cong S^{n-1}$. If $n-1 = 1$ then the proof does not work.

So now: if $\rho, \rho' : \pi_1(M) \longrightarrow PSL_2(\mathbb{C})$ are discrete & faithful rep^{n}s with $\mathbb{H}^3/\mathrm{Im}(\rho)$ & $\mathbb{H}^3/\mathrm{Im}(\rho')$ both finite-volume & complete, then ρ is conjugate to ρ'!

Def^{n} 4.14: Call this special unique-up-to-conjugacy discrete & faithful rep^{n} $\pi_1(M) \xrightarrow{\rho_0} PSL_2(\mathbb{C})$ the discrete and faithful rep^{n} (or canonical rep^{n}).

Not^{n}: Let $X_0(M)$ $([\tilde{X}_0(M)])$ be the $\underbrace{\text{irreducible}}_{\text{in the algebro-geometric sense.}}$ component of $X(M)$ containing ρ_0.

Similarly, if $z_0 \in \mathcal{S}(M,T)$ is geometric then it is unique, and $\mathrm{Hol}_{z_0} = \rho_0$.

So we have an algebraic map (given by polynomials)

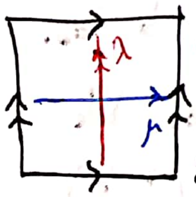
$$\mathrm{Hol} : \mathcal{S}_0(M,T) \longrightarrow X_0(M)$$

where $\mathcal{S}_0(M,T) \subseteq \mathcal{S}(M,T)$ is the irreducible component containing z_0.

§4.9 - Dehn Surgery Space

Suppose $[\rho] \in X(M)$ is a rep^{n}. The homothety part of $\rho(\alpha)$ is independent of the conjugation.

Now suppose $\partial M \cong \mathbb{T}^2$. Suppose $\mu, \lambda \in \pi_1(\partial M) \cong \mathbb{Z}^2$ generate.



∂M

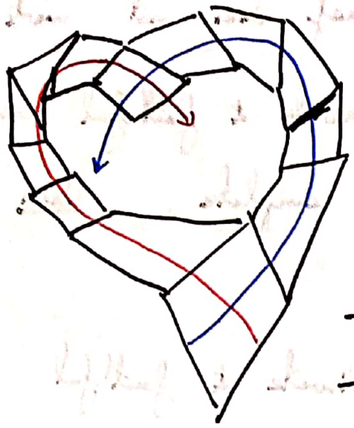
Def^{n} 4.15 If $(p,q) \in \mathbb{R}^2$ solve the DSS equation

$$p\log(h(\rho(\mu))) + q\log(h(\rho(\lambda))) = \pm 2\pi i,$$

then say (p,q) belongs to

$DSS(M)$, Dehn surgery space.

Note: $\mathbb{H}^3/\mathrm{Im}(\rho)$ has completion a manifold $\Leftrightarrow$ $(p,q)\in\mathbb{Z}^2$, p,q coprime.



$\left.\rho(\mu^p)\rho(\lambda^q)=\mathrm{Id}\right\}$ since μ,λ commute, have that $\rho(\mu)$, $\rho(\lambda)$ are simultaneously diagonalisable.

Deduce the DSS equation.

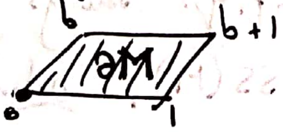
Break for movie night 2: images of DSS by SnapPy.

Thm 4.4 **Theorem** (Hyperbolic Dehn Filling Theorem [Thurston]): Suppose $(M,\mathcal{T},z_0)$ is a geometric triangulation; suppose $\partial M\cong\mathbb{T}^2$. Then $\mathrm{DSS}(M)\subseteq\mathbb{R}^2$ contains a neighbourhood of ∞.

In particular, $\exists K>0$ st. if $(p,q)\in\mathbb{Z}^2$ with $\gcd(p,q)=1$ and $p^2+q^2>K$, then the Dehn filling $M(\frac{p}{q})$ is a hyperbolic mfd (closed, complete, finite-volume.)

History: Thurston gives new examples [c.f. knots] but also a deformation theory. Thus one can use "geometric limit" arguments.

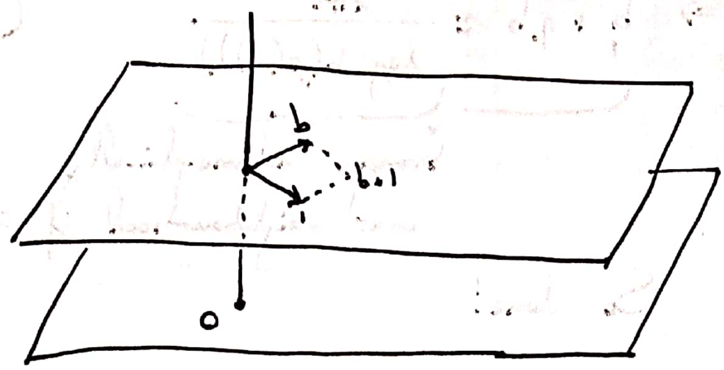
Proof: Since $\dim_{\mathbb{C}}(\mathcal{S}_0)\geq 1$, there are shapes arbitrarily close to z_0. Let $\rho=\rho_z$ be the holonomy repⁿ corresponding to z. We conjugate ρ_0 to arrange $\rho_0(\mu)=\begin{pmatrix}1&1\\0&1\end{pmatrix}$, $\rho_0(\lambda)=\begin{pmatrix}1&b\\0&1\end{pmatrix}$ with $b\notin\mathbb{R}$. Picture:



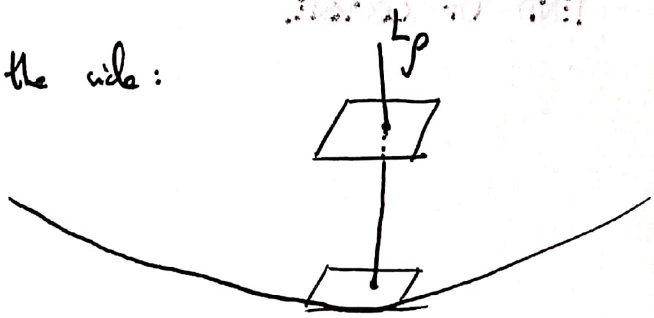
Suppose ρ is nearby. So $\rho(\mu)\approx\begin{pmatrix}1&1\\0&1\end{pmatrix}$, $\rho(\lambda)\approx\begin{pmatrix}1&b\\0&1\end{pmatrix}$ to first order.

Let L_ρ be the common invariant axis of $\rho(\mu), \rho(\lambda)$.
Note that as $\rho \neq \rho_0$ in $X_0(M)$, ρ is not the discrete & faithful rep^n. Hence $\mathbb{H}^3/\rho(\langle\mu,\lambda\rangle)$ is not complete. By an exercise we have already given, neither $\rho(\mu)$ nor $\rho(\lambda)$ is parabolic.

<u>Picture</u>:



From the side:



Let $\partial_\pm L_\rho$ be the ends of L_ρ. <u>Pick</u> one end (this is a subtle point), say $\partial_+ L_\rho$, and apply small conjugation to send $\partial_+ L_\rho$ to ∞.

So, arrange $\rho(\mu) = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha^{-1} \end{pmatrix}$, $\rho(\lambda) = \begin{pmatrix} \beta & B \\ 0 & \beta^{-1} \end{pmatrix}$ with $\alpha \approx \alpha^{-1} \approx \beta \approx \beta^{-1} \approx 1$ and $b \approx B$.

Notice that $\mathrm{Fix}(\rho(\mu)) = \{\infty, \frac{-\alpha}{\alpha^2-1}\}$ (simple computation).

Hence $\partial_- L_\rho = \frac{-\alpha}{\alpha^2-1}$.

But $\mathrm{Fix}(\rho(\mu)) = \mathrm{Fix}(\rho(\lambda))$ so

$$\begin{pmatrix} \beta & B \\ 0 & \beta^{-1} \end{pmatrix}\left(\frac{-\alpha}{\alpha^2-1}\right) = \frac{-\beta\alpha + B(\alpha^2-1)}{\beta^{-1}(\alpha^2-1)} = \frac{-\alpha}{\alpha^2-1}$$

$$\Rightarrow -\beta^2\alpha + B(\alpha^2-1)\beta = -\alpha$$

$$\Rightarrow \quad B\beta(\alpha^2-1) = \alpha(\beta^2-1)$$

$$\Rightarrow \quad B = \frac{\alpha}{\beta} \cdot \frac{\beta^2-1}{\alpha^2-1}$$

$$\Rightarrow \quad b \approx \frac{\beta^2-1}{\alpha^2-1} \approx \frac{\log(\beta^2)}{\log(\alpha^2)} = \frac{\log(h(\rho(\lambda)))}{\log(h(\rho(\mu)))}$$

$$\Rightarrow \quad \underbrace{p + qb \approx \frac{2\pi i}{\log(h(\rho(\mu)))}}_{\text{changes holomorphically over neighbourhood of } \infty \text{ in } \mathbb{C}.}$$

(all $\approx$ mean to first order).

So done! ∎

END OF COURSE.