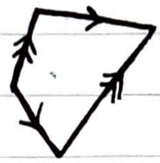
10/3/23 GEOMETRIC TOPOLOGY, SAUL SCHLEIMER

LECTURE 19

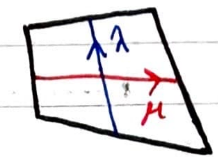
Last time: $\mathrm{Sim}(\mathbb{C})$, $\mathrm{Dil}(\mathbb{C})$, $\mathrm{Euc}(\mathbb{C})$, $\mathrm{Trans}(\mathbb{C})$

Q. Suppose $S \cong \mathbb{T}^2$ is obtained by glueing opposite sides of a quadrilateral Q.

Picture.



Pick μ, λ meridian and longitude.



Then

$h(\mu) = 1$ iff $h(\lambda) = 1$
iff S_Q is a translation surface
iff $\mathrm{Dev}: \widetilde{S_Q} \to \mathbb{C}$ is onto
iff $\mathrm{Dev}: \widetilde{S_Q} \to \mathbb{C}$ is injective
iff $\mathrm{Dev}: \widetilde{S_Q} \to \mathbb{C}$ is a homeomorphism

Q. The following three-manifolds are homeomorphic

(1) $S^3 \setminus$ a fig. eight

(2) $f: \mathbb{T}^2 \to \mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$, $f(x,y) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

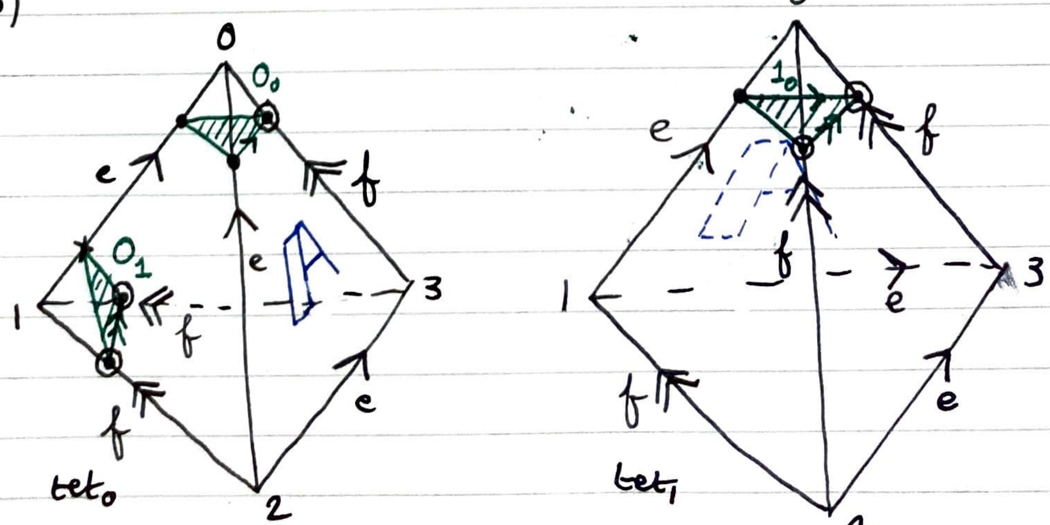
Remove zero and form

$$f^\circ: \mathbb{T}^2 \setminus \{0\} \longrightarrow \mathbb{T}^2 \setminus \{0\}$$

and get

$$M_{f^\circ} = \text{mapping torus of } f^\circ.$$

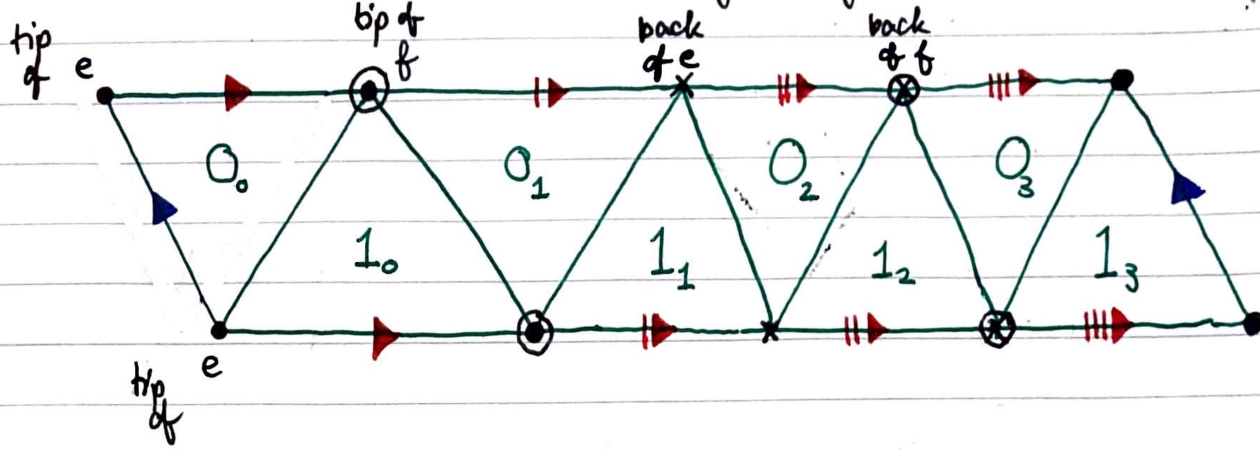
(3)



$M = T \setminus \{\text{the vertex}\}$, T the ideal triang. above.

L. The end of $M = T \setminus \{\text{vertex}\}$ is of the form $\mathbb{T}^2 \times [0,\infty)$.

Pf. We build the vertex link. (The face glueings are determined!)

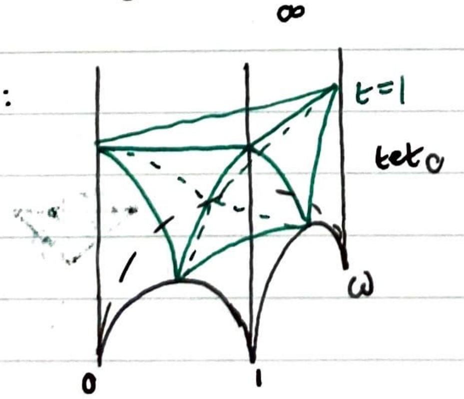


So the link of the ideal vertex is a torus, as desired!

Thm. (Thurston) Make tet 0 and tet 1 regular hyp. ideal tetrahedra (so all edge parameters are $z(e) = w = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, exercise!).
Then glueing faces by hyperbolic isometries (each unique, want orientation preserving) $M = T \setminus \{\text{vertex}\}$ receives a finite volume hyperbolic metric which is complete.

Pf. ① $\mathrm{vol}(M) = 2 \times$ vol. of reg. ideal hyperbolic tetrahedron.
$= 2v_3$.

Picture:



Cutting tet_0 along horospherical triangles (at "height" 1), we get 5 pieces.
The octahedron in the centre is compact so finite volume.
The chimnies each have volume

$$\int_1^{\infty} \frac{\frac{\sqrt{3}}{4} \cdot \frac{1}{t^2}}{t}\, dt = \frac{\sqrt{3}}{4} \int_1^{\infty} \frac{dt}{t^3} < \infty .$$

② Every point $p \in M$ lies in a tet. interior, a face interior, or an edge interior. (We removed the vertex.)

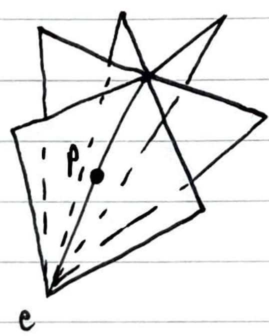
~~If p lies in a tet. or face, then build a hyperbolic 3-ball nbd. by construction.~~

If p in tet., there is a hyperbolic ball nbd. in that tet.
If p in a face, there are two $\frac{1}{2}$ nbds in adjacent tet.s.
If p in edge, there are 6 $\frac{1}{6}$ nbd.s in adjacent tet.s.

Note. Here, we use the fact that

$$\sum_{[e_i]=e} \log(z(e_i)) = 2\pi i$$

Thus, we must have a consistent choice of log for all angle parameters $z(e)$.



③ Completeness next time!