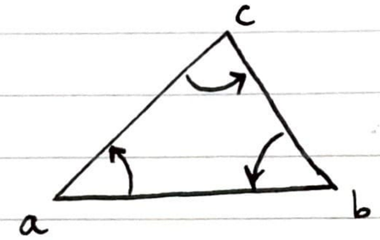
9/3/23 GEOMETRIC TOPOLOGY, SAUL SCHLEIMER

Recall: Suppose f a triangle in $\mathbb{C}$, with $f = f(a,b,c)$ a hull of a,b,c.
Assume a,b,c are ordered anti-clockwise around the boundary of f.
Picture



Define

$$z_a = \frac{c-a}{b-a}$$ (the rotation + dilation around a that takes b to c).

$$z_b = \frac{a-b}{c-b}$$

$$z_c = \frac{b-c}{a-c}$$

Q. $z_a z_b z_c = -1$, $z_b = \frac{-1}{z_a - 1}$, $z_c = \frac{z_a - 1}{z_a}$

Q. Consider the ideal
$(z_a z_b z_c + 1,\ z_b(z_a - 1) + 1, \dots) \subset \mathbb{Z}[z_a, z_b, z_c]$,
of all relations among complex angles.
What is a (minimal?) generating set for the invariant relations (invariant under the action of $z_a \to z_b \to z_c \to z_a$).

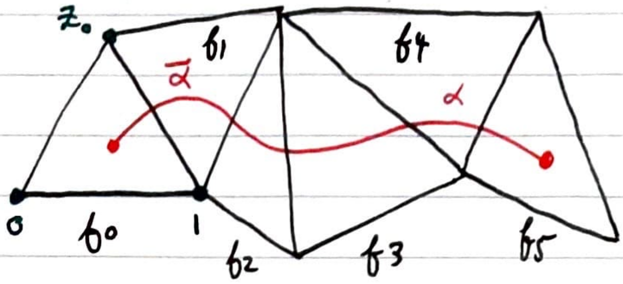
X- STRUCTURES AND THE DEVELOPING MAP

Suppose S a surface, $T = \{f_i\}$ is a triangulation,
suppose $z : \{\text{corners of faces } f\} \longrightarrow \mathbb{C} \setminus \{0\}$.
[Side hypothesis, image of z should lie in $\{w \in \mathbb{C} \mid \mathrm{Im}(w) > 0\}$.]

Set $S^\circ = S \setminus V$ (vertices of T).
We define for any $p \in S \setminus V$,
$$\mathrm{Dev}_p : \widetilde{S^\circ} \longrightarrow \mathbb{C}$$
as follows.
Fix $\tilde{q} \in \widetilde{S^\circ}$. So $\tilde{q} = [\alpha]$, where $\alpha : [0,1] \to S^\circ$ has $\alpha(0) = p$, and α is transverse to edges of T.

Let $(f_j)_{j=0}^{N}$ be the sequence of triangles visited by α.
So $p \in f_0$.

Picture.



We place f_0 in $\mathbb{C}$ with vertices a, b, c at $(0, 1, z_a)$.
We place f_1 in $\mathbb{C}$ via the unique similarity

$$\mathrm{Aff}(\mathbb{C}) = \mathrm{Sim}(\mathbb{C}) = \{az+b \mid a, b \in \mathbb{C},\ a \neq 0\}$$
$$\mathrm{Dil}(\mathbb{C}) = \{az+b \mid a \in \mathbb{R}\setminus\{0\},\ b \in \mathbb{C}\}$$
$$\mathrm{Euc}(\mathbb{C}) = \{az+b \mid a \in S^1,\ b \in \mathbb{C}\}$$
$$\mathrm{Trans}(\mathbb{C}) = \{z+b \mid b \in \mathbb{C}\}$$

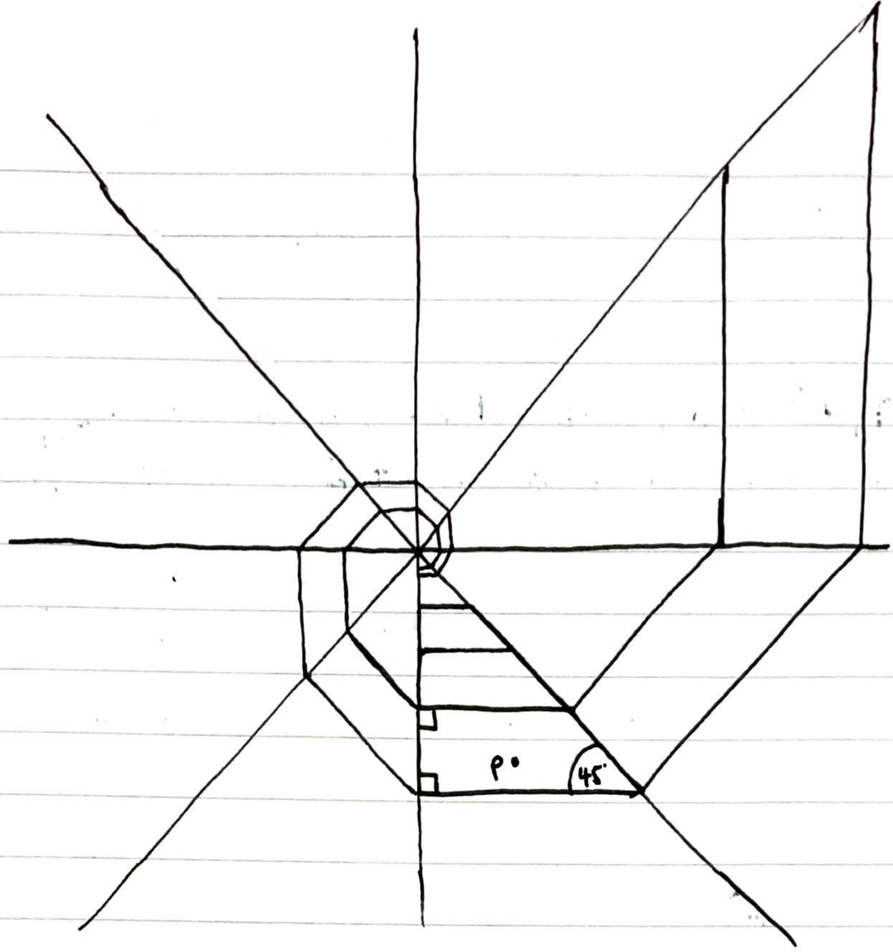
$$\mathrm{Sim} \supseteq \mathrm{Euc} \supseteq \mathrm{Trans}$$
$$\mathrm{Sim} \supseteq \mathrm{Dil} \supseteq \mathrm{Trans}$$

sending the common edge glueing f_1 to f_0 along their common edge picked out by α (they might have more than one common edge).
Do this inductively: lay out all f_j inductively.
Then $\bar{\alpha}(1)$ (where $\bar{\alpha}$ the path in $\mathbb{C}$) is the image of $\mathrm{Dev}_p(\tilde{q})$.

Eg. Suppose Q is a quadrilateral.
Draw a diagonal, and identify opposite sides of Q to get (S, T, z).

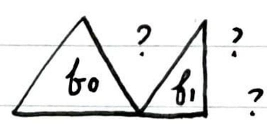
Q. Dev_p is well-defined, i.e. independent of choice of a representing $\tilde{q}$.

Point. If $[\alpha] \in \pi_1(S^\circ, p)$, then $\alpha(0) = \alpha(1) = p$, so f_N is similar to f_0
(Note the vertices are labelled.)



Q. Why delete the set V from S?

A. We do not know how to develop through vertices.



Defn. This also induces a homomorphism called holonomy,

$$\mathrm{Hol}_p : \pi_1(S^\circ, p) \longrightarrow \mathrm{Sim}(\mathbb{C}).$$

Defn. Say that (S, T, z) is a punctured X-structure if $\mathrm{Hol}_p(\pi_1)$ lies in $X \leq \mathrm{Sim}(\mathbb{C})$.

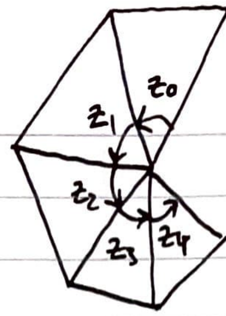
Rk. Another interesting choice of X is

$$\text{Half-Trans}(\mathbb{C}) = \{ \pm z + b \mid b \in \mathbb{C} \}.$$

UNPUNCTURING

Defn. Let $v \in T$ be a vertex. Let z_i be the adjacent complex angles.

Picture.

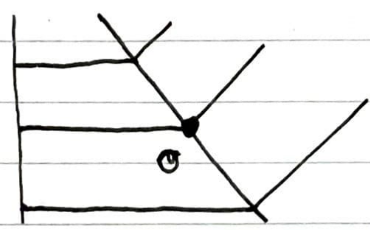


the cone angle at v is the sum

$$\sum \arg(z_i).$$

Lemma. If (S, T, Z) is a punctured dilation structure, then all cone angles lie in $2\pi\mathbb{Z}$ [and positive].

The converse is false: pick a quadrilateral Q, let S_Q be the resulting punctured similarity structure on the torus. Then there is one vertex with cone angle 2π and dilation 1:



I.e. $\sum \arg(z_i) = 2\pi$,
$\sum \log(\mathrm{dil}(z_i)) = 0$.
($\prod z_i = 1$ does not suffice.)

Defn. Suppose Q a quadrilateral. Let S_Q be the resulting punctured similarity structure on the two-torus.
Pick $\mu, \lambda \in \pi_1(\mathring{S}_Q)$ a meridian and longitude (that is, they generate $\pi_1(S_Q)$).
If $\gamma \in \mathrm{Sim}(\mathbb{C})$, define the
homothety part $h(\gamma)$,
translational part $t(\gamma)$,
via $\gamma(z) = h(\gamma)z + t(\gamma)$

Q. $h(\mu) = 1$ iff $h(\lambda) = 1$.
iff Q is a parallelogram
iff S_Q a translation surface
iff $\mathrm{Dev}_p^\bullet$ a homeomorphism, where
$\mathrm{Dev}_p^\bullet : S_Q \to \mathbb{C}$ ($\bullet$ = unpunctured)

"Unpuncturing:"

L. If $\mathrm{Hol}_p(\alpha) = \mathrm{Id}$ for all loops α about vertices, then Dev_p descends to a map $\mathrm{Dev}_p^\bullet : \widetilde{S} \to \mathbb{C}$, and Hol_p descends to a homomorphism
$\mathrm{Hol}_p^\bullet : \pi_1(\widetilde{S}) \to \mathrm{Sim}(\mathbb{C})$.