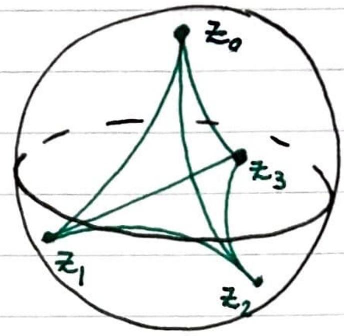
7/3/23 GEOMETRIC TOPOLOGY, SAUL SCHLEIMER

LECTURE 17

CROSS RATIOS

Suppose (z_0, z_1, z_2, z_3) are four distinct points in $\mathbb{C} \cup \{1/0\} = \hat{\mathbb{C}}$. We define the ideal tetrahedron with these points of $\hat{\mathbb{C}}$ as ideal vertices by connecting them with geodesics and taking the convex hull in $\mathbb{H}^3$.



We want to build manifolds out of such tetrahedra, and so must understand them up to the action of $\mathrm{Isom}(\mathbb{H}^3)$.
[History]

Recall $PSL(2,\mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H}^3)$ is "simply three-transitive".
So we orient our tetrahedra and send our tetrahedron w/ vert. (z_0, z_1, z_2, z_3) to one with vertices $(z, 1, 0, \infty)$.
[This ordering is used by Ahlfors, Purcell, Calegari, Snappy.]
So we use the Möbius transf. $\gamma \in PSL(2,\mathbb{C})$, where

$$\gamma(w) = \frac{w - z_2}{w - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}$$ [Scale to ensure $\gamma \in SL(2,\mathbb{C})$.]

So $$\boxed{z = \frac{z_0 - z_2}{z_0 - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}.}$$

this is the cross-ratio, denoted $(z_0, z_1; z_2, z_3)$.

Note. We made a choice - there are $24 = |S_4|$ possible choices here. That is, S_4 acts on cross-ratios via

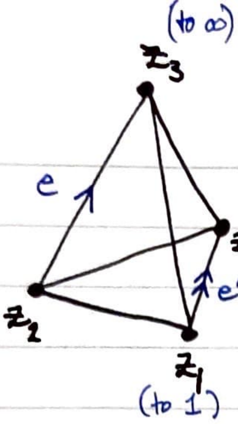
$$\sigma \cdot (z_0, z_1; z_2, z_3) = (z_{\sigma(0)}, z_{\sigma(1)}; z_{\sigma(2)}, z_{\sigma(3)}).$$

Q. $K_4 = \{ Id, (01)(23), (02)(13), (03)(12) \}$ acts trivially on cross-ratios, and explain geometrically.

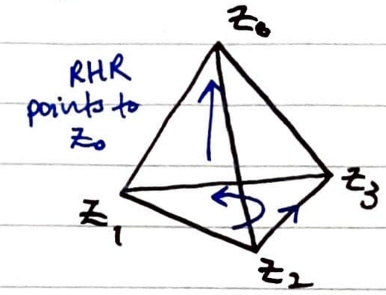
So the $\mathrm{Sym}(4)$ $(= S_4)$ action factors through

$$\mathrm{Sym}(4)/K_4 \cong D_6,$$ a dihedral group.

Note.



If t is an oriented tetrahedron, then orienting an edge e induces (via the right-hand rule) an orientation of e', the opposite edge.



We claim that $(ij) \in \mathrm{Sym}(4)$ reverses the orientation of the tetrahedron.

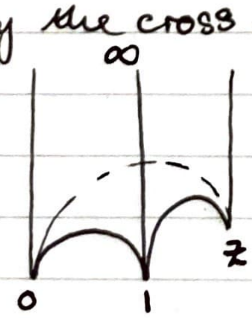
Instead we consider $C_3 < D_6$ the cyclic group.
These are represented in $\mathrm{Sym}(4)$ by three-cycles.
Using still $(z, 1; 0, \infty) = (z_0, z_1; z_2, z_3)$,

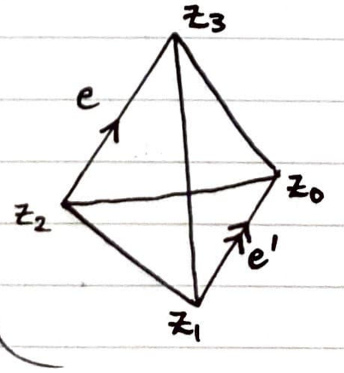
$$
\begin{aligned}
(0\,1\,2)\cdot(z_0, z_1; z_2, z_3) &= (z_1, z_2; z_0, z_3) \\
&= (1, 0; z, \infty) \\
&= \frac{1-z}{1-\infty}\cdot\frac{0-\infty}{0-z} \\
&= \frac{z-1}{z} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} z
\end{aligned}
$$

Q. $(0\,2\,1)(z, 1; 0, \infty) = \dfrac{-1}{z-1} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} z.$

EDGE LABELS

We label the oriented edges of t (hull of (z_0, z_1, z_2, z_3)) by the cross ratio of the induced oriented vertices.





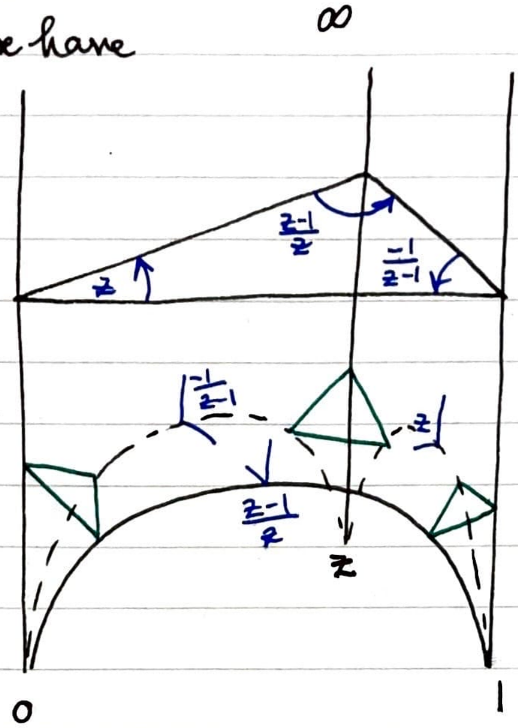
labelled as:

Define
$$z(e) = (z_0, z_1; z_2, z_3).$$

Q. $z(e) = z(e') = z(\bar{e})$. reversed edge

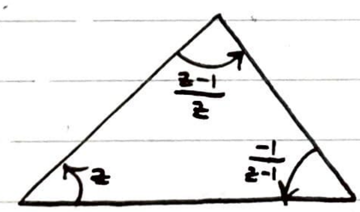
Thus we have

Q.



Q. (Q) Could also have looked at the triangles from the other vertices. They all receive the same labelling, so "all four triangles are "similar"".

Labelling the cusp triangle:
looking from ∞, we see in the cusp Δ



Note. $z \cdot \frac{z-1}{z} \cdot \frac{-1}{z-1} = -1$.

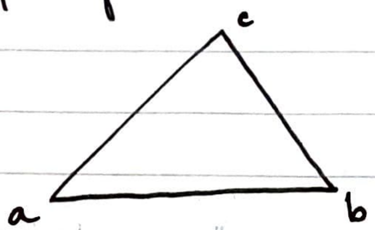
so. $\arg(z) + \arg\left(\frac{z-1}{z}\right) + \arg\left(\frac{-1}{z-1}\right) = \pi$.

$\operatorname{dil}(z) + \operatorname{dil}\left(\frac{z-1}{z}\right) + \operatorname{dil}\left(\frac{-1}{z-1}\right) = 0$.

Q. Suppose $\operatorname{Im}(z) > 0$.
Check that $\operatorname{Im}\left(\frac{-1}{z-1}\right), \operatorname{Im}\left(\frac{z-1}{z}\right) > 0$,
so we may choose a consistent branch of log to make the above true.

To understand the labelling of cusp triangles, we introduce <u>complex angles</u>.

Suppose $f \subset \mathbb{C}$ is the convex hull of $a, b, c \in \mathbb{C}$,



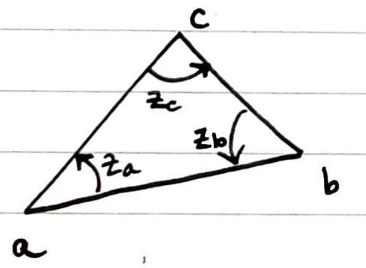
$f = f(a,b,c)$

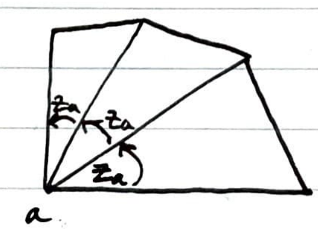
Use $\gamma \in \text{Affine}(\mathbb{C})$ to move f to $f(0,1,z_a)$.

Q. $z_a = \dfrac{c-a}{b-a}$

[and so $z_b = \dfrac{a-b}{c-b}$, $z_c = \dfrac{b-c}{a-c}$.]

Picture.





If we multiply by z_a and translate, we see a bit of a tiling.

Q. $z_b = \dfrac{-1}{z_a - 1}$, $z_c = \dfrac{z_a - 1}{z_a}$.

Claim These triangles tile about 0: