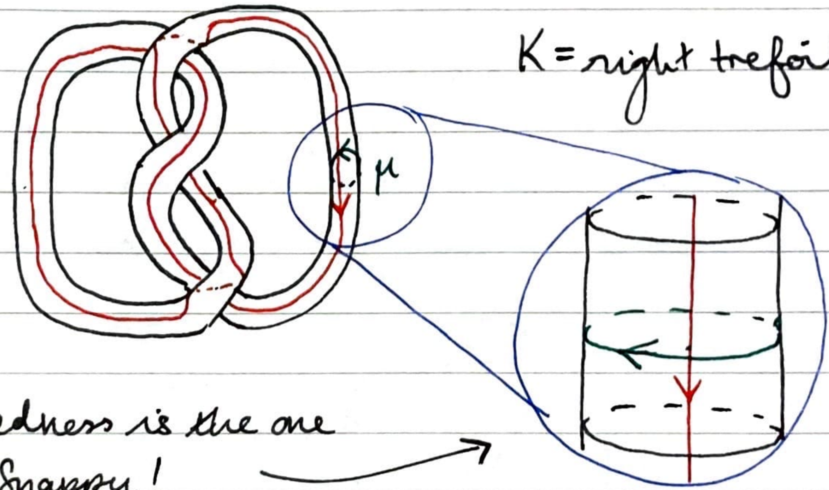
LECTURE 11

Notation. $K \subset S^3$ a (tame) knot.
$N(K) \subset S^3$ a closed product neighbourhood, $N(K) \cong D^2 \times S^1$.
$X_K = S^3 \setminus \mathrm{int}(N(K))$.
Let $T = \partial X_K = \partial N_K$.

Framing. $\mu = \partial D^2 \times \{pt\} \subset T$ is the meridian.
λ = generator of $\ker(H_1(T) \longrightarrow H_1(X_K))$
= homological longitude.

Picture.

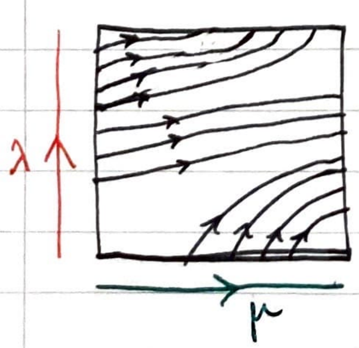


K = right trefoil.

This handedness is the one used by Snappy!
[See peripheralcurves.c]

Defn. The curve $\alpha \subset T$ so that $[\alpha] = p[\mu] + q[\lambda]$ in $H_1(T)$ is called the slope of $p/q \in \mathbb{Q} \cup \{1/0\}$.

Q. Eg.



What slope is this?

Notation. $X_K(\alpha) = X_K(p/q)$ is the manifold obtained by attaching a solid torus $\mathcal{U} = D^2 \times S^1$ along boundaries so that $\partial D^2 \times \{pt\}$ glues to α.

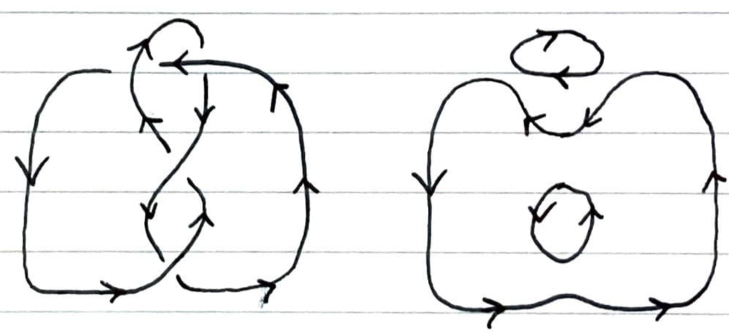
$$X_K(\alpha) = X_K \cup_\alpha \mathcal{U}$$

Notation. Suppose M, N are manifolds.
Say M is a $\begin{Bmatrix} \mathbb{Z} \\ \mathbb{Q} \end{Bmatrix}$ homology N if
$$H_k(M, \overset{\mathbb{Z}}{\mathbb{Q}}) \cong H_k(N, \overset{\mathbb{Z}}{\mathbb{Q}}) \text{ for all } k.$$

Q Suppose K is a knot in S^3.
(1) X_K is a homology solid torus.
(2) $X_K(\frac{1}{0}) \cong S^3$.
(3) $X_K(1/n)$ is an $\mathbb{Z}HS^3$ (an integer homology three-sphere).
(4) $X_K(0/1)$ is an integer homology $S^2 \times S^1$.

Language. If $X_K(p/q)$ has "property," then we call p/q a "property" slope.

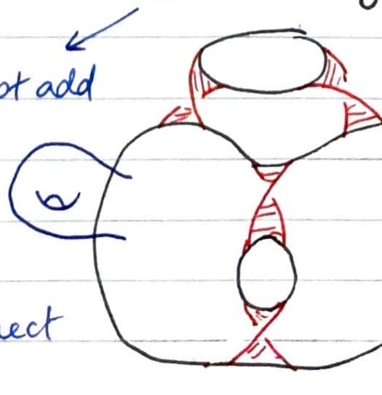
Q. (About Seifert surfaces) Show that $K \subset S^3$ has a spanning surface.
Possible solution: use Seifert's algorithm.



Step. 1 Get diagram and orient.
Step. 2 Smooth the crossings.
Step. 3 All curves bound disks in the plane. Make disjoint copies and raise to distinct heights.
Step. 4 Attach ½-twisted knots to form F. $[\partial F \cong K]$

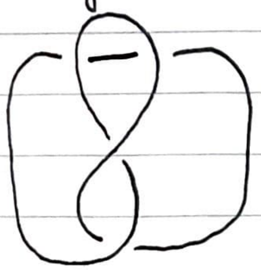
Q. Find a knot K with two distinct minimal genus spanning surfaces.

Otherwise we could just add a handle. We can always obtain a new spanning surface by stabilization, i.e. connect sum of pairs
$(S^3, F) \# (S^3, T^2)$

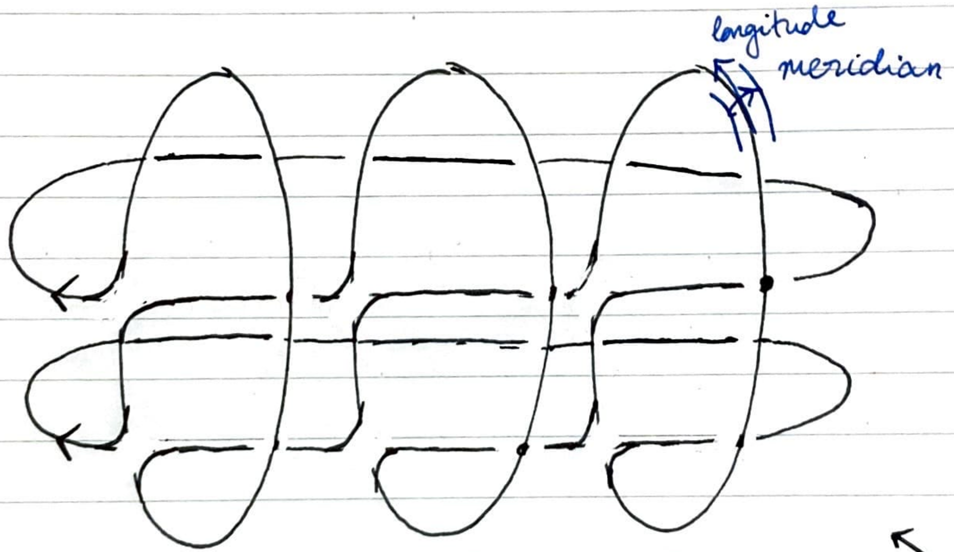


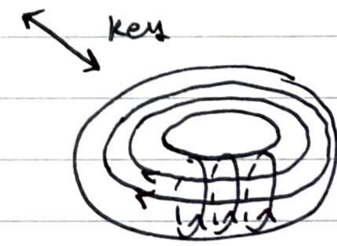
For some knots, the Seifert surface of minimal genus is not obtained by applying Seifert's algorithm to any knot diagram for the knot.

Rk. (Murphy's Law for Seifert's Algorithm)
"The algorithm need not produce a minimal spanning surface."

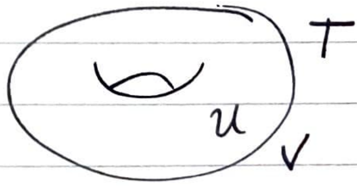


= Unknot, but Seifert's algorithm produces the minimal spanning surface.





This is a (right?) trefoil.
We get a spanning tree surface with
p disks in U
q disks in V
$p \cdot q$ $\frac{1}{4}$ - twisted bands crossing T.

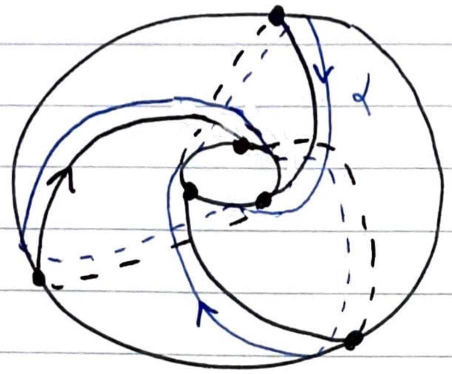


The union of the disks and bands is a surface spanning $K = K_{p,q}$, the (p,q) torus knot.

Q. This surface has genus $\frac{(p-1)(q-1)}{2}$.
Q. Draw the longitude in this picture

Note. $K_{p,q}$ lies in T. Let α be a pushoff of $K = K_{p,q}$ along $T \smallsetminus K = A$, an annulus.

Picture.



Note. μ, λ, α lie in $\partial N(K)$.
Call this the annular pushoff.

Thm. (Moser, 1971) Suppose $K = K_{p,q}$.
Suppose $r, s \in \mathbb{Z}$, $\gcd(r,s) = 1$. Set $\sigma = \det\begin{pmatrix} pq & r \\ 1 & s \end{pmatrix}$.

generic → If $|\sigma| \geq 2$, then $X_K(r/s)$ is Seifert fibred over $S^2(a,b,c)$.

special → If $|\sigma| = 1$, then $X_K(r/s) \cong L(r, sq^2)$.

really special → If $|\sigma| = 0$, then $X_K(r/s) \cong L(p,q) \# L(q,p)$ [iff $(r,s) = (pq, 1)$]

Moral. The slope α is "special"
fillings far from α are "nicer".