27/1/2 GEOMETRIC TOPOLOGY

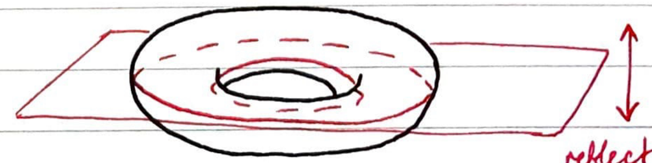
Question. How do we extend the connect sum to non-orientable manifolds?

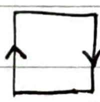
Eg. $\mathbb{RP}^2 \# \mathbb{RP}^2 \cong \mathbb{K}^2$, the Klein bottle.

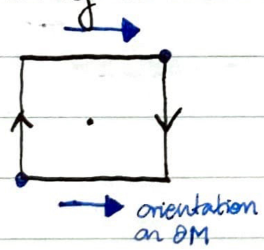
Answer. Note that $\mathbb{T}^2 \# \mathbb{T}^2 \cong S_2$, the surface of genus 2, regardless of choices, because $\mathbb{T}^2$ admits an orientation-reversing symmetry.

Defn. An orientable manifold M is achiral if M has an orientation-reversing symmetry. [Same definition for achiral pairs.]

Eg. The figure-8 knot in S^3 is achiral.
The trefoil knot is chiral.



Now: For $M^2 =$  the Möbius strip,
it suffices to find ~~an ori~~ a homeomorphism $f : M^2 \to M^2$ reversing the orientation of ∂M, in order to answer the question.

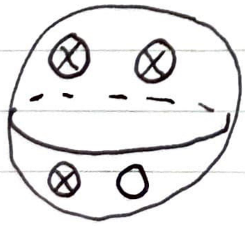


A 180° rotation about the centre.

Q. (1) $S_{g,n}$ (= genus g and n boundaries surface) is achiral.
(2) $N_{c,1}$ (defined by $N_{0,1} = S^2 D^2$ and $N_{c+1,1} = N_{c,1} \# \mathbb{RP}^2$) has symmetries that reverse the boundary.
[$N_{1,1}$ = the Möbius strip, $c = \#$ cross caps.]

In general, $N_{c,b} = \left(\#_c \mathbb{RP}^2\right) \# \left(\#_b D^2\right)$

Picture $N_{3,1}$



obtained by glueing Möbius bands to

Q. Compute $MCG(N_3)$ $\left[MCG(X) = \dfrac{\mathrm{Homeo}(X)}{\mathrm{Homeo}_0(X)} \right]$.

Q. Show that MCG (surface) is countable.
(There are theorems showing that it is also finitely-generated and finitely-presented.)

CONNECT SUMS

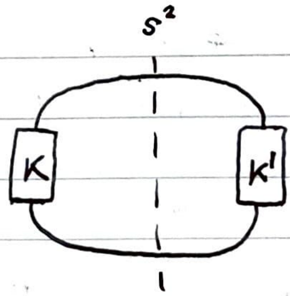
Defn. Call a manifold / manifold pair / knot prime if any connect sum decomposition is trivial, that is

if $M \cong N \# P$, then $M \cong N$ and $P \cong S^n$,
or $M \cong P$ and $N \cong S^n$.

Thm. (Jordan - Schoenflies) S^2 is prime.

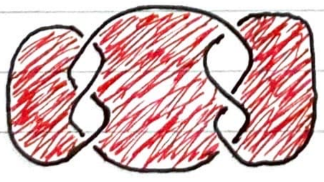
Thm. (Alexander) S^3 is prime.

Q. The unknot in S^3 is prime.
If $K \# K' \cong U$ (the unknot)
then $K \cong K' \cong U$.



Q. Suppose K, K', K'' are knots, U unknot.
(i) $K \# U \cong K$ [isotopic].
(ii) $K \# K' \cong K' \# K$.
(iii) $K \# (K' \# K'') \cong (K \# K') \# K''$.

Q. Define / recall the knot genus $g(K)$ of a knot K. [Seifert genus]
Prove that if $g(K) = 1$, then K is prime.

Eg. 

Seifert surface - has genus 2!

Thm. (Schubert, 1949) Knots have a unique factorisation into primes.

Thm. (Kneser, 1929, see also Milnor 1962) Three - manifolds have a unique factorisation into primes.

Q. (i) $S^2 \times S^1$ is prime.
(ii) $\mathbb{T}^3$ is prime.
(iii) $\mathbb{RP}^3$ is prime.

Thm. (Gordon–Luecke 1986, Waldhausen 1968)
The isomorphism type of $\pi_1(X_K)$ is a complete isotopy invariant of $K \subset S^3$ (a knot) if K is prime, up to reflection.

GEOMETRISATION VIA KNOTS

Thm. (Thurston 1982) Suppose $K \subset S^3$ is a (tame) knot.
Then K is either (i) a torus knot,
(ii) a satellite knot,
(iii) a hyperbolic knot.

Definitions:
- a torus knot $= X_k$ a Seifert fibred space
- a satellite knot $=$ X_k is toroidal
- a hyperbolic knot $=$ X_k is finite-volume hyperbolic.

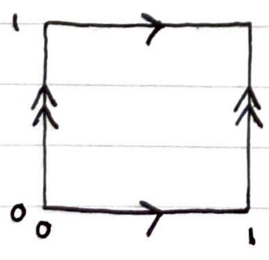
(More definitions to come...)

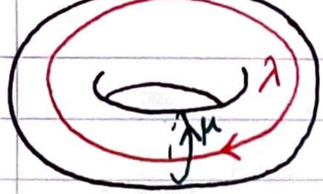
TORUS KNOTS

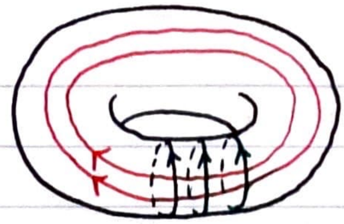
Note $S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 2\}$.
Define $C = \{(z,w) \in \mathbb{C}^2 \mid |z| = |w| = 1\}$,
$=$ standard torus

Note. $S^2 \times S^2 \longrightarrow C$
$(e^{i\theta}, e^{i\varphi}) \longmapsto (e^{i\theta}, e^{i\varphi})$ is a homeomorphism.

Picture. Use $[0,1]^2$ for some reason.

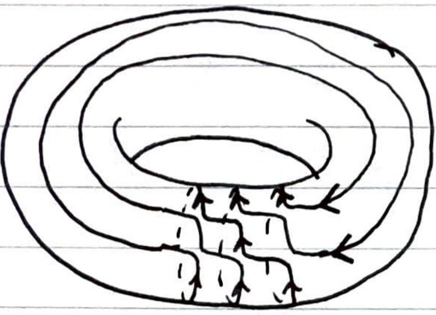






$3\mu + 2\lambda$

"adding" and "smoothing": $p\mu$ and $q\lambda$ gives a (p,q)-curve in $\mathbb{T}^2$.
("adding in homology")

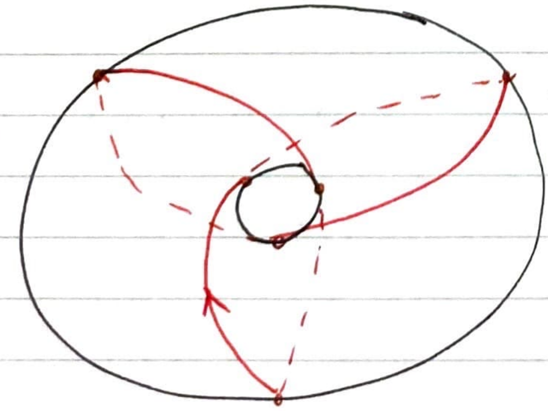


Note. $\mu \cdot \lambda = +1$,
the algebraic intersection number.

Embedding $\mathbb{T}^2$ into S^3 via C sends the (p,q)-curve to the (p,q)-torus knot:

$\mu \cdot \gamma = +2$
$\gamma \cdot \lambda = -3$

$(-3, 2)$



$(3, 2)$, the right trefoil

Q. Draw the other trefoils, e.g. $(\pm 3, \pm 2)$ and $(\pm 2, \pm 3)$.