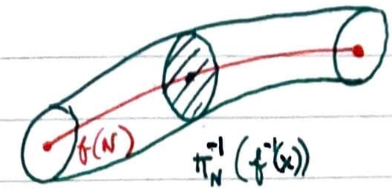
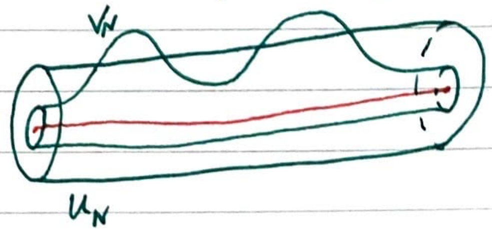
13/1/23 GEOMETRIC TOPOLOGY

L3 **Thm.** (Existence and uniqueness of regular neighbourhoods)
Suppose M^m is a manifold, and suppose $f: N^n \to M^m$ is ~~a tame embedding~~ locally flat, and $m \leq 3$.
Then there is a regular nbd. U_N of $f(N)$ in M and a bundle map $\pi_N : U_N \to N$ so that the diagram below commutes

$$\begin{tikzcd} U_N \arrow[r, hook] \arrow[d, "\pi_N"'] & M \\ N \arrow[r, "f"] & f(N) \arrow[u, phantom, "\cup"] \end{tikzcd}$$



Furthermore the regular nbd. U_N is unique (including the bundle structure) up to isotopy of M fixing $f(N)$ pointwise.



Challenge. Prove the uniqueness statement.
(The existence part is harder!)

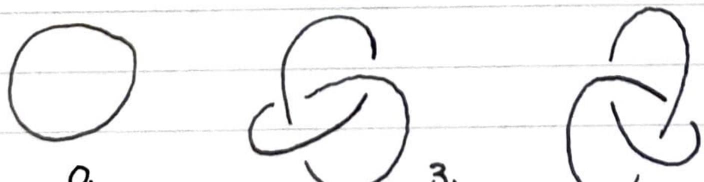
KNOTS

Knots: a tame embedding of S^1 in S^3.

Q. S^3 is the one-point compactification of $\mathbb{R}^3$.

Moral. We can draw pictures in $\mathbb{R}^3$ and pretend they're in S^3.

Eg.



0_1 3_1
[left trefoil] [right trefoil]

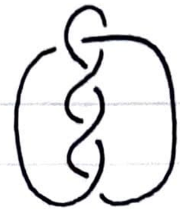
Defn 1. If K is a knot and D is a knot diagram [draw K in some plane with crossing info] then the mirror image of D yields the mirror image $\overline{K}$ of K.

Defn. 2. $\overline{K}$ is K reflected in any plane in $\mathbb{R}^3$.

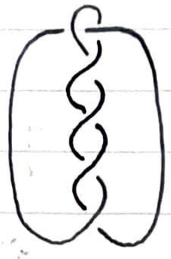
Eg.

4_1 figure 8

5_1 cinfoil

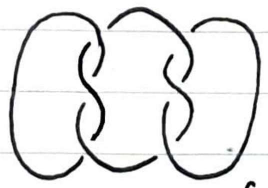


5_2 twist knot

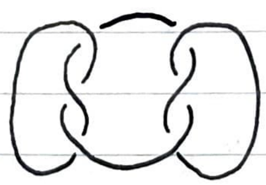


6_1

7_4

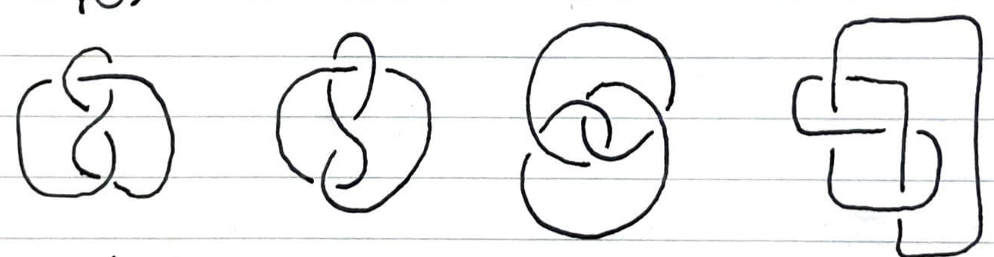


granny knot G



reef or square knot R

Eg. (Family ⑧)



Defn. Say $K, K' \subset M$ are ambi-isotopic if there is an isotopy of M taking K to K'.

Q. This is an equivalence relation.

Q. All knots in family ⑧ are isotopic.

Q. The knots from the knot tables are not isotopic.

Defn. Suppose $K \subset S^3$ is a knot. Fix U_K a regular nbd.

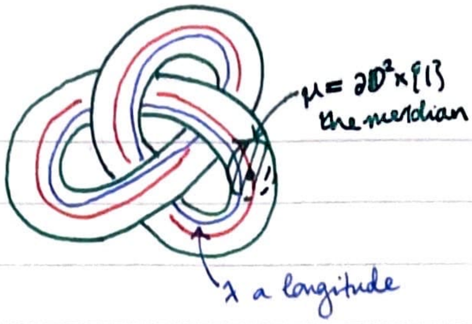
[Q.] [So $U_K \cong \mathbb{D}^2 \times S^1$.] We denote by

$$X_K = S^3 \setminus \mathring{U}_K$$

the knot exterior.

Note that $\partial X_K = \partial U_K$ is a two-torus.

We take $\mu = \partial \mathbb{D}^2 \times \{1\} \subset \partial U_K$ to be the meridian.

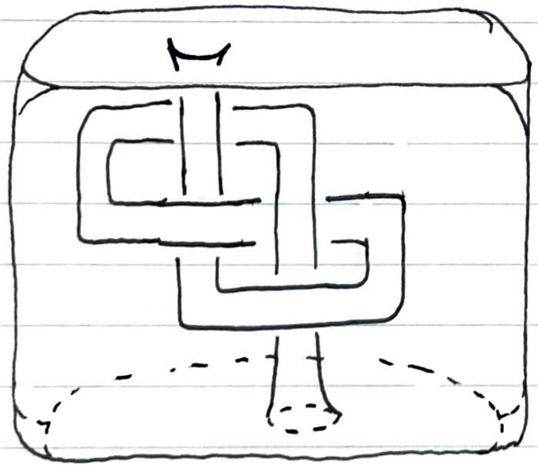


Note. Orientations are chosen so that $\mu \cdot \lambda = +1$ (the algebraic intersection number of μ, λ in ∂U_K).

Q. $S^3 \setminus K \simeq X_K$ (are homotopy equivalent).
[$S^3 \setminus K$ is called the knot complement.]

Rk. $X_G \not\cong X_R$, but $X_G \simeq X_R$
[The granny and reef knot ~~are knot~~ exteriors are not homeomorphic but are homotopic.]

Eg. (Ultra-worm in perfectly-clear block of obsidian)



This is homeo. to X_{fig8}, a 3-mf. you can hold!

Prop. If K is isotopic to K' in S^3 then $\pi_1(X_K) \cong \pi_1(X_{K'})$.

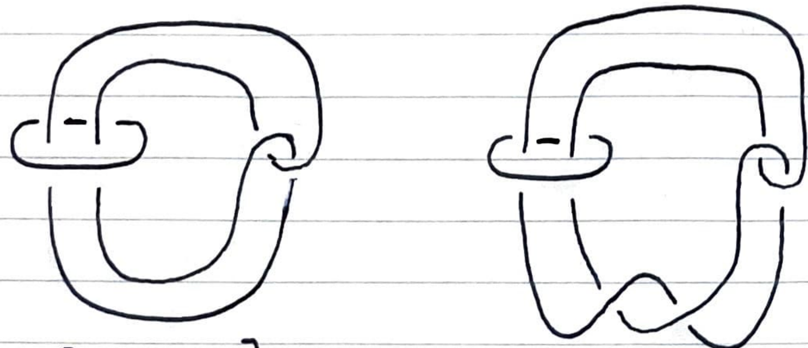
Pf. Choose U_K and $U_{K'}$ nbd.s.
Isotope U_K to $U_{K'}$ via isotopy from K to K'.
Apply uniqueness of regular nbd.s.
$\Rightarrow X_K \cong X_{K'}$ and we are done.

Cor. The isomorphism class of $\pi_1(X_K)$ is an invariant of the isotopy class of K.

Thm. [Gordon - Luecke] Suppose K, K' are knots, and $X_K \cong X_{K'}$.
Then K is isotopic to K' (or to $\overline{K}'$).

Thm. [Dehn, 1914] The right and left trefoil are not isotopic.

Picture



W [Whitehead]

TW [full twist of W].

Q. $X_W \cong X_{TW}$, but W is not isotopic to TW.