10/1/23 TOPICS IN GEOMETRIC TOPOLOGY

PLAN (1) Intro to 3-manifolds (2) lots of examples (3) sphere, disk, torus, annulus things (4) Anything! Requests!

OTHER TOPICS Hyperbolic geometry, Cannon-Thurston maps, Culler-Morgan-Shalen theory, Veering triangulations.

REFERENCES See webpage for notes by Gordon, Hatcher, Scott, Casson, Thurston.

I MANIFOLDS

Defn. M a non-empty top. space which is ① Hausdorff ② second countable ③ covered by charts to $\mathbb{R}^n$ is called an *n-manifold*.

Q. (1) If M is a manifold then its dimension is well-defined.
(2) The Hausdorfness is necessary (so give an example to show that locally $\mathbb{R}^n$ and second countable $\not\Rightarrow$ Hausdorff).

Examples. $\mathbb{R}^n$ is an n-fold.

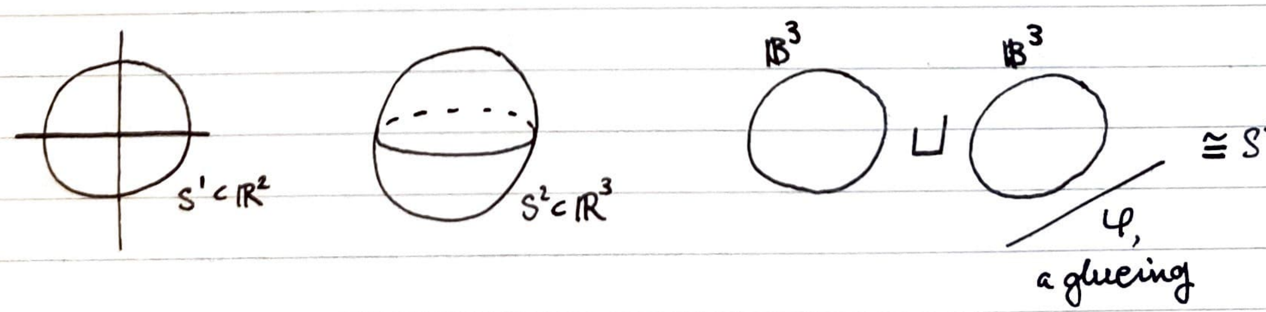
II CONSTRUCTIONS / EXAMPLES / NOTATION

A. SUBSPACES

Defn. $\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$, for $x \in \mathbb{R}^n$.

Defn. $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ is the *(unit) sphere* in $\mathbb{R}^{n+1}$.

Pictures.

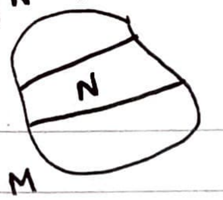


Q. Without second-countability, manifolds need not be metrisable (spelled with an 's' despite a democratic vote deciding otherwise).

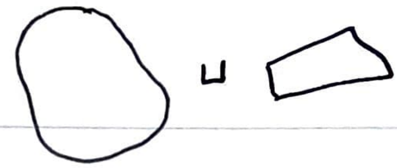
B. DISJOINT UNION

If M, N are n-manifolds, so is $M \sqcup N$.

say $N \subset M$:



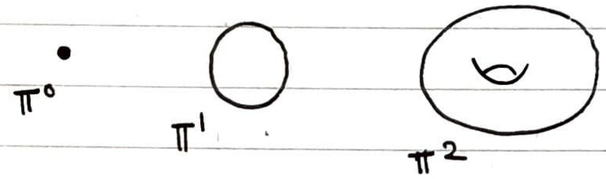
then

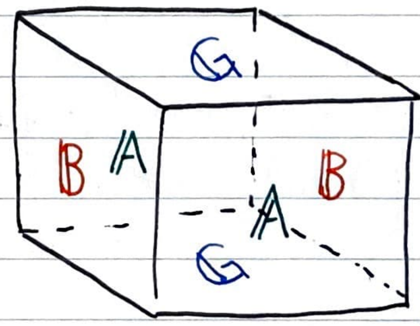
$M \sqcup N$ "=" 

C. CARTESIAN PRODUCT

Q. If M is an m-manifold and N an n-manifold, then $M \times N$ is an $(m+n)$-manifold. (Note that ϕ is not a manifold!)

Eg. Define the *n-torus* by
$\mathbb{T}^0 = \mathbb{R}^0 = \{pt.\}$, and $\mathbb{T}^{n+1} = \mathbb{T}^n \times S^1$.



$\mathbb{T}^3 =$ 

/glueing

Q. $\mathbb{T}^n \cong S^n$ for which n?
($\cong$ meaning homeomorphic)

D. QUOTIENTS (1)

Suppose G is a group acting by homeomorphisms on a space X.
We denote the quotient topological space by X/G = "X modulo G".

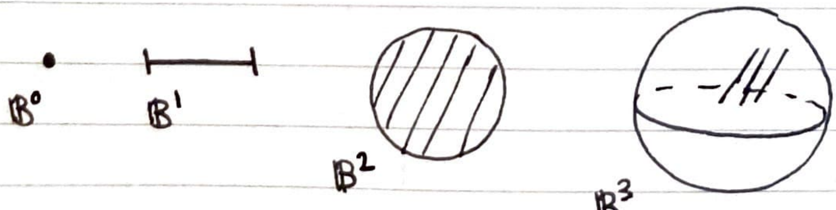
Eg. $\mathbb{Z}^n$ acts on $\mathbb{R}^n$ in the usual way, by translation.

Q. $\mathbb{R}^n/\mathbb{Z}^n \cong \mathbb{T}^n$.

QUOTIENTS (2)

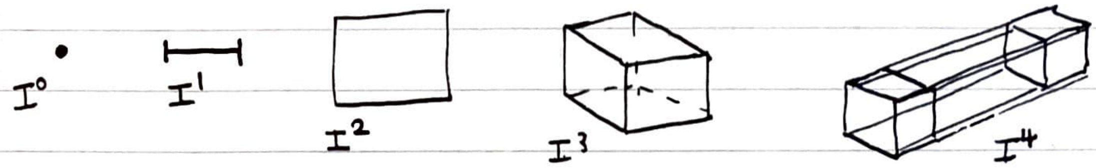
Suppose X is a top. space, $A, B \subset X$, and suppose $\varphi : A \to B$ is a bijection.
Define $X/\varphi = X/x \sim \varphi(x)$ for all $x \in A$.

Pictures.



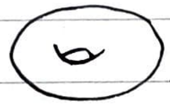
Q. $\mathbb{B}^n$ is not an n-manifold for $n > 0$.

Defn. I^n is the *n-cube*, where $I^0 = \mathbb{R}^0$, $I = [0,1]$, $I^{n+1} = I^n \times I$.



I^0 I^1 I^2 I^3 I^4

Q. Suppose M is obtained from $\mathbb{B}^3 \sqcup \mathbb{B}^3$ by glueing (i.e. quotienting) by $\varphi : S^2 \to S^2$ the identity map.
Prove that $\mathbb{B}^3 \sqcup \mathbb{B}^3 / \varphi \cong S^3$.

Q. Define the *solid torus* to be $\mathbb{B}^2 \times S^1 =$  .
Show that S^3 is a union of solid tori.

Q. $I^3 / A, B, G \cong \mathbb{T}^3$
(where $A(x,y,z) = (x, y+1, z)$, $B(x,y,z) = (x+1, y, z)$,
$G(x,y,z) = (x, y, z+1)$).

Defn. If k is a field, V a k-vector space, then
$$\mathbb{P}(V) = \frac{V \setminus \{0\}}{k \setminus \{0\}}.$$

Eg. $\mathbb{RP}^n = \dfrac{\mathbb{R}^{n+1} \setminus \{0\}}{\mathbb{R} \setminus \{0\}}$ and $\mathbb{CP}^n = \dfrac{\mathbb{C}^{n+1} \setminus \{0\}}{\mathbb{C} \setminus \{0\}}$.

Q.
(1) $\mathbb{RP}^n$ is an n-manifold.
(2) $\mathbb{CP}^n$ is a $2n$-mf.
(3) $\mathbb{RP}^1 \cong S^1$.
(4) $\mathbb{CP}^1 \cong S^2$.