Please let me know if any of the problems are unclear, have typos, or have any other mistakes.

Exercise 6.1. In this exercise, all modules are $\mathbb{Z}$–modules, and all tensor products are over $\mathbb{Z}$. Recall that if $a \in A$ and $b \in B$ are elements, then the equivalence class $a \otimes b \in A \otimes B$ is called a *pure tensor*.

- Find a pair of modules A and B and an element $c \in A \otimes B$ which is not pure.
- Find a pair of modules A and B and distinct elements $a, a' \in A$ and $b, b' \in B$ so that $a \otimes b = a' \otimes b'$.

Exercise 6.2. In this exercise, all modules are $\mathbb{Z}$–modules, and all tensor products are over $\mathbb{Z}$.

- Show that $A \otimes B \cong B \otimes A$.
- Show that $(\oplus_i A_i) \otimes B \cong \oplus_i (A_i \otimes B)$.
- Show that $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$.
- Show that $A \otimes \mathbb{Z} \cong A$.
- Show that $A \otimes \mathbb{Z}/n\mathbb{Z} \cong A/nA$.
- Show that if $f\colon A \to A'$ and $g\colon B \to B'$ are homomorphisms then there exists a unique homomorphism $f \otimes g\colon A \otimes B \to A' \otimes B'$ so that $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$.
- Show that if $\phi\colon A \times B \to C$ is bilinear then there exists a unique homomorphism $\bar{\phi}\colon A \otimes B \to C$ so that $\bar{\phi}(a \otimes b) = \phi(a, b)$.

Exercise 6.3.

1. Suppose that R is a commutative ring (with unit). Prove that $R \otimes_R R \cong R$ as R–modules.
2. Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ as $\mathbb{Z}$–modules.
3. Suppose that $R = \mathbb{Q}(\sqrt{2})$. What is $R \otimes_{\mathbb{Q}} R$ as a $\mathbb{Q}$–module?
4. [Harder.] Prove that $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}$ as $\mathbb{Z}$–modules.