Please let me know if any of the problems are unclear, have typos, or have any other mistakes.

Exercise 3.1. Suppose that F is a vector space over $\mathbb{R}$. Suppose that $\pi\colon E \to B$ is a vector bundle with fibre F. Compute the homology groups $H_*(E)$ in terms of $H_*(B)$.

Exercise 3.2. [Challenge] Here is a hands-on definition of $\mathrm{UT}(S^n)$, the unit tangent bundle to the n–sphere.

$$\mathrm{UT}(S^n) = \{(u, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : |u| = |v| = 1, \langle u, v\rangle = 0\}$$

Here $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^{n+1}$. Compute the homology groups $H_*(\mathrm{UT}(S^n))$.

Exercise 3.3. Suppose that (X, Y) is a pair of spaces and G is an abelian group. Show that there is an natural isomorphism between

$$C^k(X, Y; G) = \mathrm{Hom}(C_k(X, Y); G) \quad \text{and} \quad D^k(X, Y; G) = \ker(C^k(X; G) \to C^k(Y; G))$$

Here naturality means that if $f\colon (X, Y) \to (Z, W)$ is a map of pairs, then the resulting square commutes.

Exercise 3.4. Suppose that $(A, C), (B, D) \subset (X, Y)$ are pairs of spaces. Suppose that X is contained in the union of the interiors of A and B; similarly suppose that Y is contained in the union of the interiors of C and D. Fix G an abelian group.

We define $C_k(A+B, C+D)$ to be the cokernel of the inclusion $C_k(C+D) \to C_k(A+B)$. As usual we define $C^k(A + B, C + D; G) = \mathrm{Hom}(C_k(A + B, C + D); G)$.

- Show that $C^k(A+B, C+D; G)$ is naturally isomorphic to $D^k(A+B, C+D; G) = \ker(C^k(A + B; G) \to C^k(C + D; G))$.
- Show that $H^*(A + B, C + D; G) \cong H^*(X, Y; G)$.
- Prove the relative version of Meyer-Vietoris; that is, the following sequence of cohomology groups is exact:

$$\ldots H^k(X, Y; G) \to H^k(A, C; G) \oplus H^k(B, D; G) \to H^k(A \cap B, C \cap D; G) \to H^{k+1}(X, Y; G) \ldots$$

Exercise 3.5. Suppose that X is a space and R is a ring. Prove that the cup product on cochains is:

- R–linear in the first and second coordinates,
- associative, and
- has the constant function $1 \in C^0(X, R)$ as its identity element.

Show, by means of an example, that the cup product at the level of cochains is not skew-commutative.