Please turn in Exercises 1.1, 1.5, and 1.6 before 2pm, on 19 October (Friday), in the slot near the front office. Please let me know if any of the problems are unclear or have typos. See the class webpage for further suggestions.

`http://homepages.warwick.ac.uk/~masgar/Teach/Current/class.html`

Exercise 1.1. Suppose that R and S are commutative rings with unit. Suppose that $\phi\colon R \to S$ is a ring homomorphism. Prove that either

- $\phi(1_R) = 1_S$ or
- $\mathrm{Im}(\phi)^\times$ consists only of zero divisors.

Exercise 1.2. Suppose that F and E are fields. Suppose that $\phi\colon F \to E$ is a ring homomorphism. Prove that either

- ϕ is injective or
- ϕ is trivial: that is, the constant function with value 0_E.

Exercise 1.3. Suppose that R and S are commutative rings with unit. Suppose that $\phi\colon R \to S$ is a bijective homomorphism. Prove that ϕ^{-1} is also a homomorphism. Prove also that $\phi(1_R) = 1_S$. We call ϕ an *isomorphism*. (If $R = S$ then ϕ is called an *automorphism*.)

Exercise 1.4. Suppose that F is a field containing $\mathbb{Q}$ as a subfield. Suppose that τ is an automorphism of F. Prove that for all $r \in \mathbb{Q}$ we have $\tau(r) = r$. That is, $\mathbb{Q} \subset \mathrm{Fix}(\tau)$. Deduce that the group $\mathrm{Aut}(\mathbb{Q})$ is trivial: that is, it consists of $\mathrm{Id}_{\mathbb{Q}}$, only.

Exercise 1.5. Suppose that $p \in \mathbb{Z}$ is a prime, greater than one. Let $\lambda = \sqrt{p}$. Give a careful proof that

$$\mathbb{Q}(\lambda) = \{a + b\lambda \mid a, b \in \mathbb{Q}\}$$

is a subfield of $\mathbb{R}$.

Exercise 1.6. Suppose that $p \in \mathbb{Z}$ is a prime, greater than one. Let $\lambda = \sqrt{p}$. Give a careful proof that the group $\mathrm{Aut}(\mathbb{Q}(\lambda))$ has exactly two elements.